SECTION 1-4 Absolute Value in Equations and Inequalities

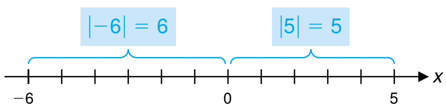
- Absolute Value and Distance
- Absolute Value in Equations and Inequalities
- Absolute Value and Radicals

This section discusses solving absolute value equations and inequalities.

• Absolute Value and Distance

We start with a geometric definition of absolute value. If a is the coordinate of a point on a real number line, then the distance from the origin to a is represented by $|a|$ and is referred to as the **absolute value** of a. Thus, $|5| = 5$, since the point with coordinate 5 is five units from the origin, and $|-6| = 6$, since the point with coordinate -6 is six units from the origin (Fig.1).

FIGURE 1 Absolute value.



Symbolically, and more formally, we define absolute value as follows:

DEFINITION 1

Absolute Value

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$|4| = 4$

$|-3| = -(-3) = 3$

[*Note:* $-x$ is positive if x is negative.]

Both the geometric and nongeometric definitions of absolute value are useful, as will be seen in the material that follows. Remember:

The absolute value of a number is never negative.

EXAMPLE 1 **Absolute Value of a Real Number**

(A) $|\pi - 3| = \pi - 3$ Since $\pi \approx 3.14$, $\pi - 3$ is positive.

(B) $|3 - \pi| = -(3 - \pi) = \pi - 3$ Since $3 - \pi$ is negative

Matched Problem 1 Write without the absolute value sign:

(A) $|8|$ (B) $|\sqrt[3]{9} - 2|$ (C) $|-\sqrt{2}|$ (D) $|2 - \sqrt[3]{9}|$

Following the same reasoning used in Example 1, the next theorem can be proved (see Problem 79 in Exercise 1-4).

Theorem 1

For all real numbers a and b,

$$|b - a| = |a - b|$$

We use this result in defining the distance between two points on a real number line.

DEFINITION 2

Distance between Points *A* and *B*

Let A and B be two points on a real number line with coordinates a and b, respectively. The **distance between *A* and *B*** is given by

$$d(A, B) = |b - a|$$

This distance is also called the **length of the line segment** joining A and B.

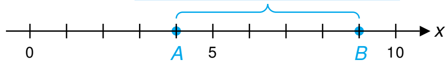
EXAMPLE 2 Distance between Points on a Number Line

Find the distance between points A and B with coordinates a and b, respectively, as given.

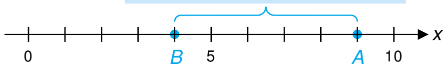
(A) $a = 4$, $b = 9$ (B) $a = 9$, $b = 4$ (C) $a = 0$, $b = 6$

Solutions

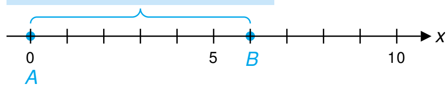
(A) $d(A, B) = |9 - 4| = |5| = 5$



(B) $d(A, B) = |4 - 9| = |-5| = 5$



(C) $d(A, B) = |6 - 0| = |6| = 6$



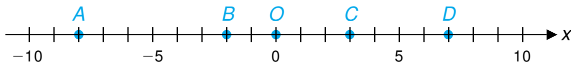
It should be clear, since $|b - a| = |a - b|$, that

$$d(A, B) = d(B, A)$$

Hence, in computing the distance between two points on a real number line, it does not matter how the two points are labeled—point A can be to the left or to the right of point B. Note also that if A is at the origin O, then

$$d(O, B) = |b - 0| = |b|$$

Matched Problem 2 Use the number line below to find the indicated distances.



(A) $d(C, D)$ (B) $d(D, C)$ (C) $d(A, B)$
(D) $d(A, C)$ (E) $d(O, A)$ (F) $d(D, A)$

• Absolute Value in Equations and Inequalities

The interplay between algebra and geometry is an important tool when working with equations and inequalities involving absolute value. For example, the algebraic statement

$$|x - 1| = 2$$

can be interpreted geometrically as stating that the distance from x to 1 is 2.

EXPLORE-DISCUSS 1 Write geometric interpretations of the following algebraic statements:

(A) $|x - 1| < 2$ (B) $0 < |x - 1| < 2$ (C) $|x - 1| > 2$

EXAMPLE 3 **Solving Absolute Value Problems Geometrically**

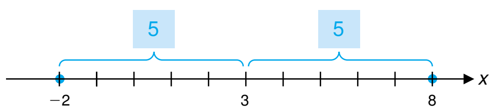
Interpret geometrically, solve, and graph. Write solutions in both inequality and interval notation, where appropriate.

(A) $|x - 3| = 5$ (B) $|x - 3| < 5$
(C) $0 < |x - 3| < 5$ (D) $|x - 3| > 5$

Solutions (A) Geometrically, $|x - 3|$ represents the distance between x and 3. Thus, in $|x - 3| = 5$, x is a number whose distance from 3 is 5. That is,

$$x = 3 \pm 5 = -2 \quad \text{or} \quad 8$$

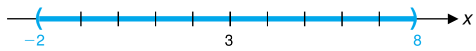
The solution set is $\{-2, 8\}$. This *is not* interval notation.



(B) Geometrically, in $|x - 3| < 5$, x is a number whose distance from 3 is less than 5; that is,

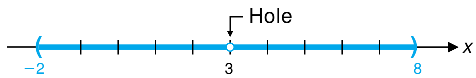
$$-2 < x < 8$$

The solution set is $(-2, 8)$. This *is* interval notation.



(C) The form $0 < |x - 3| < 5$ is frequently encountered in calculus and more advanced mathematics. Geometrically, x is a number whose distance from 3 is less than 5, but x cannot equal 3. Thus,

$$-2 < x < 8 \qquad x \neq 3 \qquad \text{or} \qquad (-2, 3) \cup (3, 8)$$



(D) Geometrically, in $|x - 3| > 5$, x is a number whose distance from 3 is greater than 5; that is,

$$x < -2 \qquad \text{or} \qquad x > 8 \qquad \text{or} \qquad (-\infty, -2) \cup (8, \infty)$$

CAUTION

Do not confuse solutions like

$$-2 < x \qquad \text{and} \qquad x < 8$$

which can also be written as

$$-2 < x < 8 \qquad \text{or} \qquad (-2, 8)$$

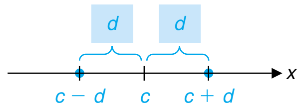
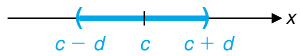
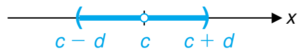
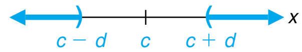
with solutions like

$$x < -2 \qquad \text{or} \qquad x > 8$$

which cannot be written as a double inequality or as a single interval.

We summarize the preceding results in Table 1.

TABLE 1 Geometric Interpretation of Absolute Value Equations and Inequalities

Form ($d > 0$)	Geometric interpretation	Solution	Graph
$\lvert x - c \rvert = d$	Distance between x and c is equal to d.	$\{c - d, c + d\}$	
$\lvert x - c \rvert < d$	Distance between x and c is less than d.	$(c - d, c + d)$	
$0 < \lvert x - c \rvert < d$	Distance between x and c is less than d, but $x \neq c$.	$(c - d, c) \cup (c, c + d)$	
$\lvert x - c \rvert > d$	Distance between x and c is greater than d.	$(-\infty, c - d) \cup (c + d, \infty)$	

Matched Problem 3 Interpret geometrically, solve, and graph. Write solutions in both inequality and interval notation, where appropriate.

(A) $|x + 2| = 6$ (B) $|x + 2| < 6$
(C) $0 < |x + 2| < 6$ (D) $|x + 2| > 6$

[*Hint:* $|x + 2| = |x - (-2)|$.]

EXPLORE-DISCUSS 2

Describe the set of numbers that satisfies each of the following:

(A) $2 > x > 1$ (B) $2 > x < 1$
(C) $2 < x > 1$ (D) $2 < x < 1$

Explain why it is never necessary to use double inequalities with inequality symbols pointing in different directions. Standard mathematical notation requires that all inequality symbols in an expression must point in the same direction.

Reasoning geometrically as before (noting that $|x| = |x - 0|$) leads to Theorem 2.