An Introduction to Proofs and the Mathematical Vernacular [1]

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Dedicated to the memory of my mother:
Coralyn S. Day, November 6, 1922 – May 13, 2008.

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Preface: To the Student

In the standard year (or two) of university calculus and differential equations courses you have learned a lot of mathematical techniques for solving various types of problems. Along the way you were offered "proofs" of many of the fundamental relationships and formulas (stated as "theorems"). Perhaps occasionally you were asked to "show" or "prove" something yourself as a homework problem. For the most part, however, you probably viewed the proofs as something to be endured in the lectures and skimmed over in the book. The main emphasis of those courses was on learning *how to use* the techniques of calculus, and the proofs may not have seemed very helpful for that.

Historically, techniques of calculation were the principal concern of mathematics. But as those techniques became more complex, the concepts behind them became increasingly important. You are now at the stage of your mathematical education where the focus of your studies shifts from techniques to ideas. The goal of this book is to help you make the transition from being a mere user of mathematics to becoming conversant in the language of mathematical discussion. This means learning to critically read and evaluate mathematical statements and being able to write mathematical explanations in clear, logically precise language. We will focus especially on mathematical proofs, which are nothing but carefully prepared expressions of mathematical reasoning.

By focusing on how proofs work and how they are expressed we will be learning to think about mathematics as mathematicians do. This means learning the language and notation (symbols) which we use to express our reasoning precisely. But writing a proof is always preceded by *finding* the logical argument that the proof expresses, and that may involve some exploration and experimentation, trying various ideas, and being creative. We will do some practicing with mathematics that is familiar to you, but it is important to practice with material that you don't already know as well, so that you can really have a try at the creative exploration part of writing a proof. For that purpose I have tried to include some topics that you haven't seen before (and may not see elsewhere in the usual undergraduate curriculum). On the other hand I don't want this course to be divorced from what you have learned in your previous courses, so I also have tried to include some problems that ask you to use your calculus background. Of course the book includes many proofs which are meant to serve as examples as you learn to write your own proofs. But there are also some which are more involved than anything you will be asked to do. Don't tune these out — learning to read a more complicated argument written by someone else is also a goal of this course.

Some consider mathematics to be a dry, cold (even painful) subject. It certainly is very difficult in places. But it can also be exciting when we see ideas come together in unexpected ways, and see the creative ways that great minds have exploited unseen connections between topics. To that end I have included a few examples of really clever proofs of famous theorems. It is somewhat remarkable that a subject with such high and objective standards of logical correctness should at the same time provide such opportunity for the expression of playful genius. This is what attracts many people to the study of mathematics, particularly those of us who have made it our life's work. I hope this book communicates at least a little of that excitement to you.

Observe that theorems, lemmas, and propositions are numbered consecutively within chapters. For instance in Chapter 4 we find Theorem 4.6 followed by Lemma 4.9. Many theorems also have names (e.g. "The Division Theorem"). You can locate them using the index. Definitions are not numbered, but are listed in the index by topic. Problems are numbered within chapters similarly; for instance you will find Problem 2.8 in Chapter 2. The end of each problem statement is marked by a dotted line followed by a cryptic word in a box, like this:

.... `whatever`

The word in the box is of no significance to you; it is just a reminder for me of the computer file which contains the text of the problem.

References are marked by a number in brackets like this: [9]. You can find the full reference for [9] in the Bibliography at the back of the book. (This is the customary format for citing references in mathematics.) Sometimes a page number or other information is included after the number to help you find the relevant place in the reference, like [9, page 99].

It is unlikely that every problem in the book will be assigned. If a problem you are assigned refers to another which was not assigned, you are *not* expected to work the unassigned problem, but can just take its statement for granted and use it in your solution of the assigned problem. For instance, if you were assigned Problem 4.17 (which asks you to use Problem 4.16), you are free to use the fact stated in Problem 4.16 whether or not it was assigned.

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