Introduction to functions

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A function is a rule which operates on one number to give another number. However, not every rule describes a valid function. This unit explains how to see whether a given rule describes a valid function, and introduces some of the mathematical terms associated with functions.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- recognise when a rule describes a valid function,
- be able to plot the graph of a part of a function,
- find a suitable domain for a function, and find the corresponding range.

Contents

1. What is a function?

Here is a definition of a function.

A function is a rule which maps a number to another unique number.

In other words, if we start off with an input, and we apply the function, we get an output.

For example, we might have a function that added 3 to any number. So if we apply this function to the number 2, we get the number 5. If we apply this function to the number 8, we get the number 11. If we apply this function to the number x, we get the number $x + 3$.

We can show this mathematically by writing

$$f(x) = x + 3.$$

The number x that we use for the input of the function is called the *argument* of the function. So if we choose an argument of 2, we get

$$f(2) = 2 + 3 = 5.$$

If we choose an argument of 8, we get

$$f(8) = 8 + 3 = 11.$$

If we choose an argument of -6, we get

$$f(-6) = -6 + 3 = -3.$$

If we choose an argument of z, we get

$$f(z) = z + 3.$$

If we choose an argument of x^2, we get

$$f(x^2) = x^2 + 3.$$

At first sight, it seems that we can pick any number we choose for the argument. However, that is not the case, as we shall see later. But because we do have some choice in the number we can pick, we call the argument the *independent variable*. The output of the function, e.g. $f(x)$, $f(5)$, etc. depends upon the argument, and so this is called the *dependent variable*.

Key Point

A function is a rule that maps a number to another unique number.

The input to the function is called the *independent variable*, and is also called the *argument* of the function. The output of the function is called the *dependent variable*.

2. Plotting the graph of a function

If we have a function given by a formula, we can try to plot its graph. Suppose, for example, that we have a function f defined by

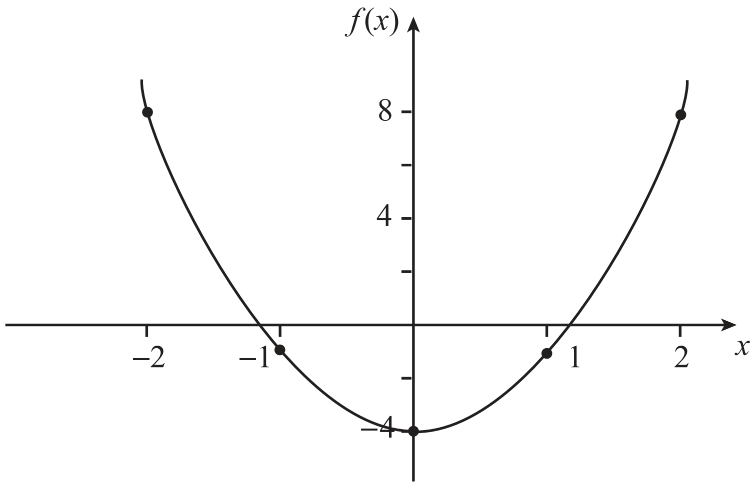
$$f(x) = 3x^2 - 4.$$

The argument of the function (the independent variable) is x, and the output (the dependent variable) is $3x^2 - 4$. So we can calculate the output of the function for different arguments:

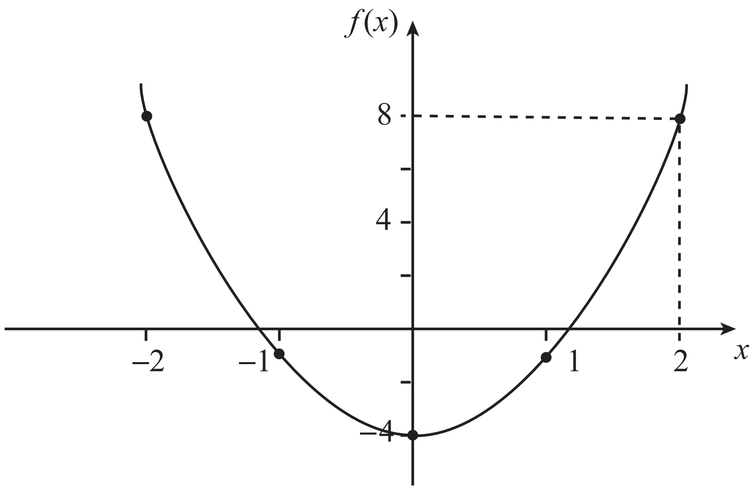
$$\begin{aligned} f(0) &= 3 \times 0^2 - 4 &&= -4 \\ f(1) &= 3 \times 1^2 - 4 &&= -1 \\ f(2) &= 3 \times 2^2 - 4 &&= 8 \\ f(-1) &= 3 \times (-1)^2 - 4 &&= -1 \\ f(-2) &= 3 \times (-2)^2 - 4 &&= 8. \end{aligned}$$

We can put this information into a table to help us plot the graph of the function.

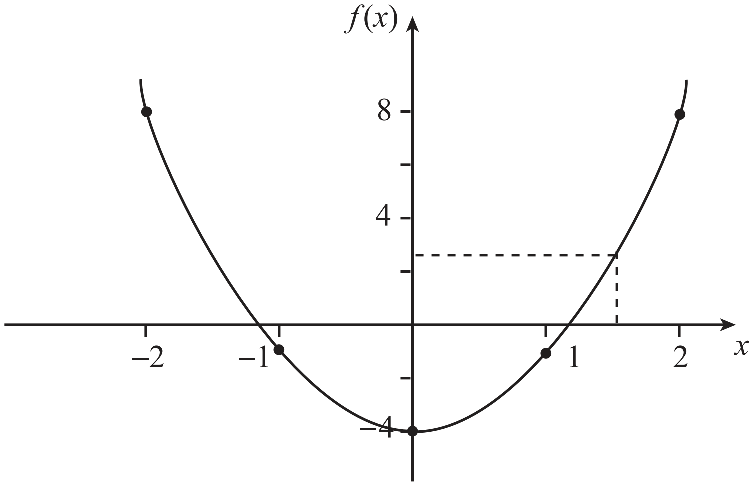
x	$f(x)$
-2	8
-1	-1
0	-4
1	-1
2	8



We can use the graph of the function to find the output corresponding to a given argument. For instance, if we have an argument of 2, we start on the horizontal axis at the point where $x = 2$, and we follow the line up until we reach the graph. Then we follow the line across so that we can read off the value of $f(x)$ on the vertical axis. In this case, the value of $f(x)$ is 8. Of course we already know this, because $x = 2$ is one of the values in our table.



But we can also use the graph for values of x which are not in our table. If we have an argument of 1.5, we follow the line up to the graph, and then across to the vertical axis. The result is a number between 2 and 3.



If we want to calculate this number exactly, we can substitute 1.5 into the formula:

$$f(1.5) = 2.75$$

3. When is a function valid?

Our definition of a function says that it is a rule mapping a number to another unique number. So we cannot have a function which gives two different outputs for the same argument. One easy way to check this is from the graph of the function, by using a ruler. If the ruler is aligned vertically, then it only ever crosses the graph once; no more and no less. This means that the graph represents a valid function.

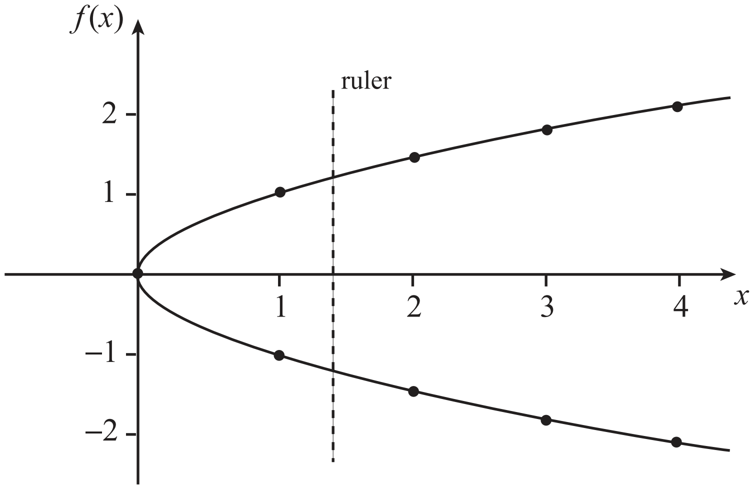
What happens if we try to define a function with more than one output for the same argument? Let's try an example. Suppose we try to define a function by saying that

$$f(x) = \sqrt{x}.$$

In the same way as before, we can produce a table of results to help us plot the graph of the function:

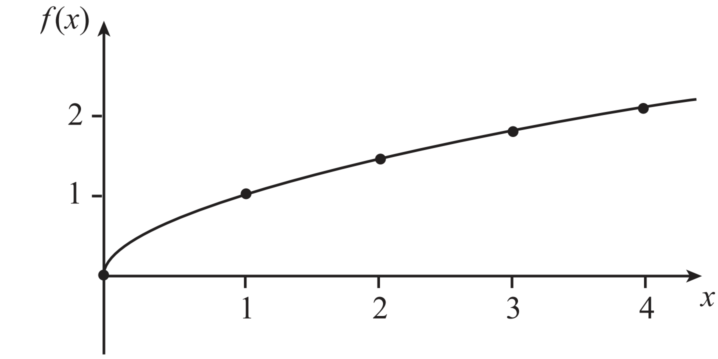
$$\begin{aligned} f(0) &= 0 \\ f(1) &= \pm 1 \\ f(2) &= \pm 1.4 && \text{to 1 d.p.} \\ f(3) &= \pm 1.7 && \text{to 1 d.p.} \\ f(4) &= \pm 2. \end{aligned}$$

(If we try to use any negative arguments, we end up in trouble because we are trying to find the square root of a negative number.) Plotting the results from the table, we get the following graph.



Using the ruler, it is quite clear that there are two values for all of the positive arguments. So as it stands, this is not a valid function.

One way around this problem is to define $\sqrt{x}$ to take only the positive values, or zero: this is sometimes called the *positive square root* of x. However, there is still the issue that we cannot choose a negative argument. So we should also choose to restrict the choice of argument to positive values, or zero.



When considering these kinds of restrictions, it is important to use the right mathematical language. We say that the set of possible inputs is called the *domain* of the function, and the set of corresponding outputs is called the *range*. In the example above, we have defined the function as follows:

$$f(x) = \sqrt{x} \qquad x \geq 0, \quad f(x) \geq 0,$$

so that the domain of the function is the set of numbers $x \geq 0$, and the range is the corresponding set of numbers $f(x) \geq 0$.

Key Point

The *domain* of a function is the set of possible inputs. The *range* of a function is the set of corresponding outputs.

