Chapter 4A - Introduction to Functions

Definition of Function

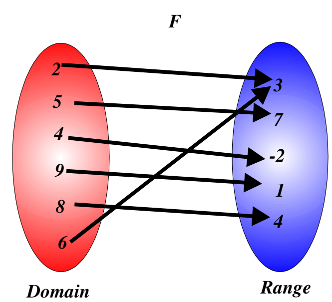
In almost every aspect of our lives, we find examples of situations where one quantity depends on another. For example, the wind chill depends on the speed of the wind; the area of a circle depends on its radius; and the amount earned by your investment depends on the interest rate.

Definitions:

- A **function** is a relation in which no two different ordered pairs have the same first element.
$$F = \{(2,3),\ (5,7),\ (4,-2),\ (9,1),\ (8,4),\ (6,3)\}$$
- A **function** can also be defined as a rule that assigns exactly one element in set B to each element in set A.
- Set A, the set of all first coordinates, is called the domain of the function.
- Set B, the set of all second coordinates, is called the range.

In the function F above, the domain is the set of first elements: $\{2,5,4,9,8,6\}$ and the range is the set of second elements: $\{3,7,-2,1,4\}$.

The function F is depicted pictorially below:



The characteristic that distinguishes a function from any other relation is that there is only one output (y-value) for each input (x-value).

Example 1: Which of the following relations are functions?

a) $\{(2,3),(-1,3),(5,3),(8,3),(0,3)\}$

b)

x	y
-3	4
-2	3
-1	2
0	1
1	0

c) $\{(1,5),(2,7)(3,9),(1,4)\}$

Solution:

a) Function. Although each output is the same, 3, there is only one output for each input so the relation is a function.

b) Function. Each x is paired with only one y.

c) Not a function. 1 is paired with two outputs: 5 and 4. We have two ordered pairs with the same first element.

Example 2: Many functions are programmed into your calculator. Use the $\sqrt{\ }$ function to find the outputs for the following inputs.

Input	$\sqrt{\ }$	Output
25		
9		
0		
$\frac{1}{4}$		
–4		

a) What output did you get for –4?

b) List 4 numbers that are in the domain of $y = \sqrt{x}$ and 4 numbers in its range.

Solution:

Input	$\sqrt{\ }$	Output
25	$\sqrt{25}$	5
9	$\sqrt{9}$	3
0	$\sqrt{0}$	0
$\frac{1}{4}$	$\sqrt{\frac{1}{4}}$	$\frac{1}{2}$
–4	$\sqrt{-4}$	*ERROR*

a) ERROR on your calculator means that $\sqrt{-4}$ is not a real number. Therefore, –4 is not in the domain of the function.

b) Answers will vary. Domain will be numbers you key into your calculator (input). Range will be the output numbers that the calculator provides on your screen.

Example 3: Which of the following represent functions? Determine the domain and range of each set.

a) $\{(2,5),\ (-1,6),\ (5,7),\ (8,5),\ (2,4)\}$

b) $\{(95,A),\ (82,B),\ (92,A),\ (62,D),\ (74,C),\ (85,B),\ (58,F),\ (77,C)\}$

c)

x	y
–1	5
0	3
1	–1
2	–3
3	–5

d)

Input	6	2	1	3	6
Output	7	9	4	7	5

Solution:

a) Not a function. 2 is paired with both 5 and 4. Domain: $\{2,-1,5,8\}$ Range: $\{5,6,7,4\}$

b) Function. Domain: $\{95,82,92,62,74,85,58,77\}$ Range: $\{A,B,C,D,F\}$

c) Function. Domain: $\{-1,0,1,2,3\}$ Range: $\{5,3,-1,-3,-5\}$

d) Not a function. 6 is paired with both 7 and 5. Domain: $\{6,2,1,3\}$ Range: $\{7,9,4,5\}$

Function Notation

To communicate functional information more effectively, we often use the symbol $f(x)$ read "f of x" instead of y to give the output for the input x. That is, $y = f(x)$. The use of $f(x)$ instead of y allows to state that f is a function and it gives us the input value x as well.

Input x	Function Rule: $f(x) = 3x - 1$	Output $f(x)$	Ordered Pair $(x, f(x))$
5	$f(5) = 3(5) - 1$	14	$(5, 14)$
3	$f(3) = 3(3) - 1$	8	$(3, 8)$
0	$f(0) = 3(0) - 1$	-1	$(0, -1)$

$$\begin{array}{l} \textit{input} = 5 \\ \downarrow \end{array}$$

We write that $\underbrace{f(5)} = 14$, meaning that the output (y) is 14 when the input (x) is 5 .

$$\begin{array}{l} \uparrow \\ \textit{output} = 14 \end{array}$$

Question: Which of the following are true about the statement $f(3) = 8$?

a) $f \cdot 3 = 8 \;\Rightarrow\; f = \frac{8}{3}$

b) 3 is the input.

c) $f(3)$ is the output.

d) 8 is the output.

e) $(3, 8)$ is an ordered pair of the function f.

Answer:

a) False. The symbol $f(x)$ does NOT mean multiply f *times* x.

b) True.

c) True.

d) True.

e)True.

Evaluating Functions

To evaluate the function f, we find the output for a given input. Consider the function $f(x) = 2x^2 - x + 1$. To evaluate $f(2)$, we plug $x = 2$ into f to find the functional value (y-value).

$$f(\ x\ ) = 2 \cdot x^2 - x + 1$$
$$\updownarrow \qquad \updownarrow \quad \updownarrow$$
$$f(\ 2\ ) = 2 \cdot 2^2 - 2 + 1 = 2(4) - 2 + 1 = 7 \quad \Rightarrow \quad f(2) = 7$$

Example 1: For $f(x) = 2x^2 - x + 1$, find a) $f(-3)$, b) $f(a+h)$, c) $f(-x)$.

Solution:

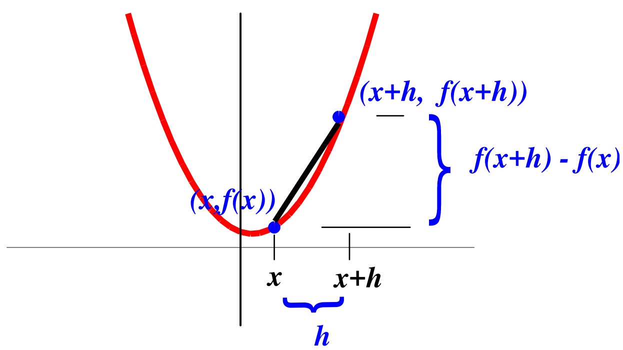
a) Replace x with -3: $f(-3) = 2(-3)^2 - (-3) + 1 = 22$

b) Replace x with the quantity $(a+h)$:

$$f(a+h) = 2(a+h)^2 - (a+h) + 1 = 2(a^2 + 2ah + h^2) - (a+h) + 1$$
$$= 2a^2 + 4ah + 2h^2 - a - h + 1$$

c) $f(-x) = 2(-x)^2 - (-x) + 1 = 2x^2 + x + 1$

An important expression in mathematics is the **difference quotient**. Graphically, the difference quotient is the slope of the line that goes through two particular points on the graph of a function. For example, the slope of the black line that connects the points $(x, f(x))$ and $(x+h, f(x+h))$ for the graph of $f(x) = 2x^2 - x + 1$ is shown below. Note that the **slope** of the line $m = \frac{f(x+h) - f(x)}{h}$ is the **difference quotient**.



Algebraically, the difference quotient is the average rate of change of the y-values with respect to their x-values. : $\frac{f(x+h) - f(x)}{h}$, $h \neq 0$.

Although the mathematical expression may seem a bit complicated at first, it is just a set of instructions written in mathematical symbols. To evaluate the difference quotient for a given function just follow the directions given in the formula:

1) Evaluate $f(x+h)$ for the function. You did this easily above.
2) Subtract the original function $f(x)$ from your result. Remember to distribute the "$-$" sign.
3) Divide the result by h and simplify.

Example 2: Evaluate $\frac{f(x+h)-f(x)}{h}$, $h \neq 0$ for the above function $f(x) = 2x^2 - x + 1$.

Solution:

$$\frac{f(x+h)-f(x)}{h} = \frac{[2(x+h)^2-(x+h)+1]-[2x^2-x+1]}{h}$$
$$= \frac{[2x^2+4xh+2h^2-x-h+1]-[2x^2-x+1]}{h}$$
$$= \frac{2x^2+4xh+2h^2-x-h+1-2x^2+x-1}{h}$$
$$= \frac{4xh+2h^2-h}{h} = 4x+2h-1$$

IMPORTANT NOTE: $f(x+h) \neq 2x^2 - x + 1 + h.$

To avoid this mistake, write the function with ( ) in place of x. Then write the input expression inside the ( ).

For example, to find $f(x+h)$ for the function $f(x) = 3x^4 - 5x^2 + 6$:write
$f(x+h) = 3(\quad)^4 - 5(\quad)^2 + 6.$

Fill in the parentheses with the input $(x+h)$ to get $f(x+h) = 3(x+h)^4 - 5(x+h)^2 + 6.$

Example 3: Evaluate $f(x) = 2x + 5$ for each of the following:

a) $f(-5)$ b) $f(-c)$ c) $f(a+h)$ d) $\frac{f(a+h)-f(a)}{h}$, $h \neq 0$

Solution:

a) $f(-5) = 2(-5) + 5 = -5$

b) $f(-c) = 2(-c) + 5 = -2c + 5$

c) $f(a+h) = 2(a+h) + 5 = 2a + 2h + 5$

d) $\frac{f(a+h)-f(a)}{h} = \frac{(2(a+h)+5)-(2a+5)}{h} = \frac{2a+2h+5-2a-5}{h} = \frac{2h}{h} = 2$

Example 4: Evaluate the difference quotient $\frac{f(x+h)-f(x)}{h}$, $h \neq 0$ for each of the following:

a) $f(x) = -x + 3$ b) $f(x) = 2x + 5$ c) $f(x) = x^2 + x - 4$ d) $f(x) = -x^2 - x - 1$

Solution:

a) $\frac{f(x+h)-f(x)}{h}$, $h \neq 0$

$$\frac{[-(x+h)+3]-[-x+3]}{h} = \frac{-x-h+3+x-3}{h} = \frac{-h}{h} = -1$$

b) $\frac{[2(x+h)+5]-[2x+5]}{h} = \frac{[2x+2h+5]-[2x+5]}{h} = \frac{2h}{h} = 2$

c) $\frac{\left[(x+h)^2+(x+h)-4\right]-[x^2+x-4]}{h} = \frac{x^2+2xh+h^2+x+h-4-x^2-x+4}{h} =$
$\frac{2xh+h^2+h}{h} = 2x+h+1$

d) $\frac{\left[-(x+h)^2-(x+h)-1\right]-[-x^2-x-1]}{h} = \frac{-(x^2+2xh+h^2)-x-h+1+x^2+x+1}{h} =$
$\frac{-x^2-2xh-h^2-x-h+1+x^2+x+1}{h} = \frac{-2xh-h^2-h}{h} = -2x-h-1$