An Introduction to Real Analysis

John K. Hunter[1]

Department of Mathematics, University of California at Davis

[1]The author was supported in part by the NSF. Thanks to Janko Gravner for a number of corrections and comments.

Abstract. These are some notes on introductory real analysis. They cover the properties of the real numbers, sequences and series of real numbers, limits of functions, continuity, differentiability, sequences and series of functions, and Riemann integration. They don't include multi-variable calculus or contain any problem sets. Optional sections are starred.

© John K. Hunter, 2014

Contents

Chapter 1. Sets and Functions 1
1.1. Sets 1
1.2. Functions 5
1.3. Composition and inverses of functions 7
1.4. Indexed sets 8
1.5. Relations 11
1.6. Countable and uncountable sets 14

Chapter 2. Numbers 21
2.1. Integers 22
2.2. Rational numbers 23
2.3. Real numbers: algebraic properties 25
2.4. Real numbers: ordering properties 26
2.5. The supremum and infimum 27
2.6. Real numbers: completeness 29
2.7. Properties of the supremum and infimum 31

Chapter 3. Sequences 35
3.1. The absolute value 35
3.2. Sequences 36
3.3. Convergence and limits 39
3.4. Properties of limits 43
3.5. Monotone sequences 45
3.6. The lim sup and lim inf 48
3.7. Cauchy sequences 54
3.8. Subsequences 55

3.9. The Bolzano-Weierstrass theorem 57

Chapter 4. Series 59
4.1. Convergence of series 59
4.2. The Cauchy condition 62
4.3. Absolutely convergent series 64
4.4. The comparison test 66
4.5. * The Riemann ζ-function 68
4.6. The ratio and root tests 69
4.7. Alternating series 71
4.8. Rearrangements 73
4.9. The Cauchy product 77
4.10. * Double series 78
4.11. * The irrationality of e 86

Chapter 5. Topology of the Real Numbers 89
5.1. Open sets 89
5.2. Closed sets 92
5.3. Compact sets 95
5.4. Connected sets 102
5.5. * The Cantor set 104

Chapter 6. Limits of Functions 109
6.1. Limits 109
6.2. Left, right, and infinite limits 114
6.3. Properties of limits 117

Chapter 7. Continuous Functions 121
7.1. Continuity 121
7.2. Properties of continuous functions 125
7.3. Uniform continuity 127
7.4. Continuous functions and open sets 129
7.5. Continuous functions on compact sets 131
7.6. The intermediate value theorem 133
7.7. Monotonic functions 136

Chapter 8. Differentiable Functions 139
8.1. The derivative 139
8.2. Properties of the derivative 145
8.3. The chain rule 147
8.4. Extreme values 150
8.5. The mean value theorem 152

8.6. Taylor's theorem 154
8.7. * The inverse function theorem 157
8.8. * L'Hôspital's rule 162

Chapter 9. Sequences and Series of Functions 167
9.1. Pointwise convergence 167
9.2. Uniform convergence 169
9.3. Cauchy condition for uniform convergence 170
9.4. Properties of uniform convergence 171
9.5. Series 175

Chapter 10. Power Series 181
10.1. Introduction 181
10.2. Radius of convergence 182
10.3. Examples of power series 184
10.4. Algebraic operations on power series 188
10.5. Differentiation of power series 193
10.6. The exponential function 195
10.7. * Smooth versus analytic functions 197

Chapter 11. The Riemann Integral 205
11.1. The supremum and infimum of functions 206
11.2. Definition of the integral 208
11.3. The Cauchy criterion for integrability 215
11.4. Continuous and monotonic functions 219
11.5. Linearity, monotonicity, and additivity 222
11.6. Further existence results 230
11.7. * Riemann sums 234
11.8. * The Lebesgue criterion 238

Chapter 12. Properties and Applications of the Integral 241
12.1. The fundamental theorem of calculus 241
12.2. Consequences of the fundamental theorem 246
12.3. Integrals and sequences of functions 251
12.4. Improper Riemann integrals 255
12.5. * Principal value integrals 261
12.6. The integral test for series 265
12.7. Taylor's theorem with integral remainder 268

Chapter 13. Metric, Normed, and Topological Spaces 271
13.1. Metric spaces 271
13.2. Normed spaces 276

Click here to download full PDF material