Basics of Polynomials

A *polynomial* is what we call any function that is defined by an equation of the form

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

where $a_n, a_{n-1}, ...a_1, a_0 \in \mathbb{R}$.

Examples. The following three functions are examples of polynomials.

- $p(x) = -2x^2 - \pi x + \sqrt[2]{2}$ is a polynomial. We could rewrite $p(x)$ as $p(x) = (-2)x^2 + (-\pi)x + (\sqrt[2]{2})$, so $a_2 = -2$, $a_1 = -\pi$, and $a_0 = \sqrt[2]{2}$.

- $p(x) = 3x^4 - \frac{1}{2}x$ is a polynomial. Notice that $p(x) = (3)x^4 + (0)x^3 + (0)x^2 + (-\frac{1}{2})x + (0)$, so $a_4 = 3$, $a_3 = 0$, $a_2 = 0$, $a_1 = -\frac{1}{2}$, and $a_0 = 0$.

- $p(x) = 15$ is a polynomial. Here $a_0 = 15$.

Coefficients, degree, and leading terms

The numbers $a_n, a_{n-1}, ...a_1, a_0 \in \mathbb{R}$ in the definition of a polynomial are called the *coefficients* of the polynomial. A coefficient a_k is called the *degree k coefficient*.

We almost always write a polynomial as $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ where $a_n \neq 0$. That means we never really write a polynomial as $0x^2 + 3x$, although it is technically a polynomial. It would just be silly to write $0x^2 + 3x$ instead of $3x$, since they are equal.

If we write a polynomial as $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ where $a_n \neq 0$, then n is the *degree* of $p(x)$, a_n is the *leading coefficient* of $p(x)$, and a_nx^n is the *leading term* of $p(x)$.

For example, the leading coefficient of $4x^3 - 5x^2 + 6x - 7$ is 4. Its degree is 3, and its leading term is $4x^3$. Notice that the leading term of $4x^3 - 5x^2 + 6x - 7$ records both the leading coefficient and the degree of $4x^3 - 5x^2 + 6x - 7$.

If the degree of a polynomial is small, there is usually a word to describe it. For example, degree 0 polynomials are called constant polynomials. Degree 1 polynomials are called linear polynomials...

degree	common name	example	leading coefficient	leading term
0	constant	2	2	2
1	linear	$3x - 1$	3	$3x$
2	quadratic	$x^2 + 2x - 4$	1	x^2
3	cubic	$-x^3 - x$	-1	$-x^3$
4	quartic	$\frac{1}{2}x^4 - x^3 + 1$	$\frac{1}{2}$	$\frac{1}{2}x^4$
5	quintic	$-3x^5 + x^2 - 12$	-3	$-3x^5$

Watch out for this trick. The leading coefficient of $3+2x^5$ is 2, its leading term is $2x^5$, and its degree is 5. Such a polynomial is most often written as $2x^5+3$, but changing the way it's written does not change its degree, leading coefficient, or leading term.

The degree is the biggest exponent above any of the x's. If the degree of a polynomial equals n, then the leading coefficient is the coefficient in front of x^n, whether it's the first number written on the left of the polynomial or not.

Addition, subtraction, and multiplication

Adding, subtracting, and multiplying polynomials usually boils down to an exercise in using the distributive law.

Examples in addition and subtraction. To add or subtract the polynimials

$$p(x) = 3x^5 - 7x^3 + 4x^2 - 2$$

and

$$q(x) = x^4 - 2x^3 + 3x^2 - 5$$

you just have to pair up the coefficients whose degree is the same, and then add or subtract each pair.

Thus, $p(x) + q(x)$ equals

$$\begin{array}{lllllll} & \Big[\quad (3)x^5 + & (0)x^4 + & (-7)x^3 + & (4)x^2 + & (0)x + & (-2)\Big] \\ + & \Big[\quad (0)x^5 + & (1)x^4 + & (-2)x^3 + & (3)x^2 + & (0)x + & (-5)\Big] \end{array}$$

$$=(3+0)x^5 + (0+1)x^4 + (-7+(-2))x^3 + (4+3)x^2 + (0+0)x + (-2+(-5))$$

$$=3x^5 + x^4 - 9x^3 + 7x^2 - 7$$

and $p(x) - q(x)$ equals

$$\begin{array}{lllllll} & \Big[\quad (3)x^5 + & + & (-7)x^3 + & (4)x^2 + & + & (-2)\Big] \\ - & \Big[\quad & (1)x^4 + & (-2)x^3 + & (3)x^2 + & + & (-5)\Big] \end{array}$$

$$=(3-0)x^5 + (0-1)x^4 + (-7-(-2))x^3 + (4-3)x^2 + \qquad +(-2-(-5))$$

$$=3x^5 - x^4 - 5x^3 + x^2 + 3$$

Below are three examples of polynomial multiplication.

- $2(x-4) = 2x - 2(4) = 2x - 8$

- $$\begin{aligned} 4x^2(x^3 + 7x - 2) &= 4x^2x^3 + 4x^27x - 4x^22 \\ &= 4x^5 + 28x^3 - 8x^2 \end{aligned}$$

- $$\begin{aligned} (3x^2 + 4x)(x^4 - 2x^3 + 5) &= (3x^2 + 4x)x^4 - (3x^2 + 4x)2x^3 + (3x^2 + 4x)5 \\ &= 3x^2x^4 + 4xx^4 - 3x^22x^3 - 4x2x^3 + 3x^25 + 4x5 \\ &= 3x^6 + 4x^5 - 6x^5 - 8x^4 + 15x^2 + 20x \\ &= 3x^6 - 2x^5 - 8x^4 + 15x^2 + 20x \end{aligned}$$

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Leading term of a product is the product of leading terms

Notice that in each of the three examples above, the leading term of the product is the product of the leading terms. That is, the leading term of $2(x-4)$ is the product of 2 and x. The leading term of $4x^2(x^3+7x-2)$ is $4x^5 = (4x^2)(x^3)$, and the leading term of $(3x^2+4x)(x^4-2x^3+5)$ is $3x^6 = (3x^2)(x^4)$.

These examples illustrate an important feature of polynomial multiplication: If you multiply some polynomials together, no matter how many polynomials, you can find the leading term of the resulting product by multiplying together the leading terms of the polynomials that you started with.

Examples.

- The leading term of $2x^2-5x$ is $2x^2$. The leading term of $-7x+4$ is $-7x$. So the leading term of $(2x^2-5x)(-7x+4)$ will be $(2x^2)(-7x) = -14x^3$.
- The leading term of $5(x-2)(x+3)(x^2+3x-7)$ will be $(5)(x)(x)(x^2) = 5x^4$.
- The leading term of $(2x^3-7)(x^5-3x+5)(x-1)(5x^7+6x-9)$ equals $(2x^3)(x^5)(x)(5x^7) = 10x^{16}$.

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Exercises

For #1-5, find the degree, leading coefficient, and leading term of the given polynomials.

1.) $2x^3 - x^2 + 7x - 4$

2.) $-x^2 + \pi x$

3.) $x - 7$

4.) $23x^5 - 100 + 3x^{17}$

5.) $-4x + 2$

For #6-16, add, subtract, or multiply the polynomials as indicated in the problem.

6.) $(2x + 3) + (-x + 5)$

7.) $(3x^2 - x + 6) - (3x^2 + x - 6)$

8.) $(8x^2 - 5x - 2) + (4x^5 - x^2 + 3x + 7)$

9.) $(7x^{100} + x - 3) - (7x^{100} + 17x + 10)$

10.) $(-x^2 + 4x + 2) - (5x^4 - x^2 + 2x - 8)$

11.) $3x^2(x + 7)$

12.) $-5x(2x - 3)$

13.) $6x^2(3x + 1)$

14.) $2x(x^3 + 4x - 6)$

15.) $(x^2 + 6)(x - 5)$

16.) $(5x^3 + 8)(x^2 + 2x - 1)$