MISCELLANEOUS TOPICS IN FIRST YEAR MATHEMATICS

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Chapter 2

POLYNOMIALS

2.1. Introduction

We shall be considering polynomials with rational, real or complex coefficients. Accordingly, throughout this chapter, $\mathbb{F}$ denotes $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

DEFINITION. We denote by $\mathbb{F}[x]$ the set of all polynomials of the form

$$p(x) = p_k x^k + p_{k-1} x^{k-1} + \ldots + p_1 x + p_0, \quad \text{where } k \in \mathbb{N} \cup \{0\} \text{ and } p_0, \ldots, p_k \in \mathbb{F};$$

in other words, $\mathbb{F}[x]$ denotes the set of all polynomials in variable x and with coefficients in $\mathbb{F}$. Suppose further that $p_k \neq 0$. Then p_k is called the leading coefficient of the polynomial $p(x)$, $p_k x^k$ is called the leading term of the polynomial $p(x)$, and k is called the degree of the polynomial $p(x)$. In this case, we write $k = \deg p(x)$. Furthermore, if $p_k = 1$, then the polynomial $p(x)$ is called monic.

EXAMPLE 2.1.1. The polynomial $3x^2 + 2$ is in $\mathbb{Q}[x]$, $\mathbb{R}[x]$ and $\mathbb{C}[x]$. Furthermore, it has degree 2 and leading coefficient 3.

EXAMPLE 2.1.2. The polynomial $4x^2 + 3x - \sqrt{5}$ is in $\mathbb{R}[x]$ and $\mathbb{C}[x]$ but not $\mathbb{Q}[x]$. Furthermore, it has degree 2 and leading coefficient 4.

EXAMPLE 2.1.3. The polynomial $x^3 + (3 + 2\mathrm{i})x - 3$ is in $\mathbb{C}[x]$ but not $\mathbb{Q}[x]$ or $\mathbb{R}[x]$. Furthermore, it has degree 3 and is monic.

EXAMPLE 2.1.4. The constant polynomial 5 is in $\mathbb{Q}[x]$, $\mathbb{R}[x]$ and $\mathbb{C}[x]$. Furthermore, it has degree 0 and leading coefficient 5.

REMARK. We have defined the degree of any non-zero constant polynomial to be 0. Note, however, that we have not defined the degree of the constant polynomial 0. The reason for this will become clear from Proposition 2A.

DEFINITION. Suppose that

$$p(x) = p_k x^k + p_{k-1} x^{k-1} + \ldots + p_1 x + p_0 \qquad \text{and} \qquad q(x) = q_m x^m + q_{m-1} x^{m-1} + \ldots + q_1 x + q_0$$

are two polynomials in $\mathbb{F}[x]$. Then we write

$$p(x) + q(x) = (p_n + q_n)x^n + (p_{n-1} + q_{n-1})x^{n-1} + \ldots + (p_1 + q_1)x + (p_0 + q_0), \tag{1}$$

where $n = \max\{k, m\}$. Furthermore, we write

$$p(x)q(x) = r_{k+m} x^{k+m} + r_{k+m-1} x^{k+m-1} + \ldots + r_1 x + r_0, \tag{2}$$

where, for every $s = 0, 1, \ldots, k+m$,

$$r_s = \sum_{j=0}^{s} p_j q_{s-j}. \tag{3}$$

Here, we adopt the convention $p_j = 0$ for every $j > k$ and $q_j = 0$ for every $j > m$.

EXAMPLE 2.1.5. Suppose that $p(x) = 3x^2 + 2$ and $q(x) = x^3 + (3 + 2\mathrm{i})x - 3$. Note that $k = 2$, $p_0 = 2$, $p_1 = 0$ and $p_2 = 3$. Note also that $m = 3$, $q_0 = -3$, $q_1 = 3 + 2\mathrm{i}$, $q_2 = 0$ and $q_3 = 1$. If we adopt the convention, then $k + m = 5$ and $p_3 = p_4 = p_5 = q_4 = q_5 = 0$. Now

$$p(x) + q(x) = (0 + 1)x^3 + (3 + 0)x^2 + (0 + 3 + 2\mathrm{i})x + (2 - 3) = x^3 + 3x^2 + (3 + 2\mathrm{i})x - 1.$$

On the other hand,

$$\begin{aligned}
r_5 &= p_0q_5 + p_1q_4 + p_2q_3 + p_3q_2 + p_4q_1 + p_5q_0 = p_2q_3 = 3,\\
r_4 &= p_0q_4 + p_1q_3 + p_2q_2 + p_3q_1 + p_4q_0 = p_1q_3 + p_2q_2 = 0 + 0 = 0,\\
r_3 &= p_0q_3 + p_1q_2 + p_2q_1 + p_3q_0 = p_0q_3 + p_1q_2 + p_2q_1 = 2 + 0 + 3(3 + 2\mathrm{i}) = 11 + 6\mathrm{i},\\
r_2 &= p_0q_2 + p_1q_1 + p_2q_0 = 0 + 0 - 9 = -9,\\
r_1 &= p_0q_1 + p_1q_0 = 2(3 + 2\mathrm{i}) + 0 = 6 + 4\mathrm{i},\\
r_0 &= p_0q_0 = -6,
\end{aligned}$$

so that

$$p(x)q(x) = 3x^5 + (11 + 6\mathrm{i})x^3 - 9x^2 + (6 + 4\mathrm{i})x - 6.$$

Note that our technique for multiplication is really just a more formal version of the usual technique involving distribution, as

$$\begin{aligned}
p(x)q(x) &= (3x^2 + 2)(x^3 + (3 + 2\mathrm{i})x - 3)\\
&= (3x^2 + 2)x^3 + (3x^2 + 2)(3 + 2\mathrm{i})x - 3(3x^2 + 2)\\
&= (3x^5 + 2x^3) + (3(3 + 2\mathrm{i})x^3 + 2(3 + 2\mathrm{i})x) - (9x^2 + 6)\\
&= 3x^5 + (11 + 6\mathrm{i})x^3 - 9x^2 + (6 + 4\mathrm{i})x - 6.
\end{aligned}$$

More formally, we have

$$\begin{aligned}
&p(x)q(x) = (p_2x^2 + p_1x + p_0)(q_3x^3 + q_2x^2 + q_1x + q_0)\\
&= (p_2x^2 + p_1x + p_0)q_3x^3 + (p_2x^2 + p_1x + p_0)q_2x^2 + (p_2x^2 + p_1x + p_0)q_1x + (p_2x^2 + p_1x + p_0)q_0\\
&= (p_2q_3x^5 + p_1q_3x^4 + p_0q_3x^3) + (p_2q_2x^4 + p_1q_2x^3 + p_0q_2x^2) + (p_2q_1x^3 + p_1q_1x^2 + p_0q_1x)\\
&\quad + (p_2q_0x^2 + p_1q_0x + p_0q_0)\\
&= p_2q_3x^5 + (p_1q_3 + p_2q_2)x^4 + (p_0q_3 + p_1q_2 + p_2q_1)x^3 + (p_0q_2 + p_1q_1 + p_2q_0)x^2 + (p_0q_1 + p_1q_0)x + p_0q_0.
\end{aligned}$$

The following result follows immediately from the definitions. For purely technical reasons, we define $\deg 0 = -\infty$, where 0 represents the constant zero polynomial.

PROPOSITION 2A. *Suppose that*

$$p(x) = p_k x^k + p_{k-1}x^{k-1} + \ldots + p_1 x + p_0 \qquad \text{and} \qquad q(x) = q_m x^m + q_{m-1}x^{m-1} + \ldots + q_1 x + q_0$$

are two polynomials in $\mathbb{F}[x]$*, where both* p_k *and* q_m *are non-zero, so that* $\deg p(x) = k$ *and* $\deg q(x) = m$*. Then*
(a) $\deg p(x)q(x) = k + m$*; and*
(b) $\deg(p(x) + q(x)) \le \max\{k, m\}$*.*

PROOF. (a) Suppose first of all that $p(x)$ and $q(x)$ are both different from the zero polynomial 0. Then it follows from (2) and (3) that the leading term of the polynomial $p(x)q(x)$ is $r_{k+m}x^{k+m}$, where

$$r_{k+m} = p_0 q_{k+m} + \ldots + p_{k-1}q_{m+1} + p_k q_m + p_{k+1}q_{m-1} + \ldots + p_{k+m}q_0 = p_k q_m \ne 0.$$

Hence $\deg p(x)q(x) = k + m$. If $p(x)$ is the zero polynomial, then $p(x)q(x)$ is also the zero polynomial. Note now that $\deg p(x)q(x) = -\infty = -\infty + \deg q(x) = \deg p(x) + \deg q(x)$. A similar argument applies if $q(x)$ is the zero polynomial.

(b) Recall (1) and that $n = \max\{k, m\}$. If $p_n + q_n \ne 0$, then $\deg(p(x) + q(x)) = n = \max\{k, m\}$. If $p_n + q_n = 0$ and $p(x) + q(x)$ is non-zero, then there is a largest $j < n$ such that $p_j + q_j \ne 0$, so that $\deg(p(x) + q(x)) = j < n = \max\{k, m\}$. Finally, if $p_n + q_n = 0$ and $p(x) + q(x)$ is the zero polynomial 0, then $\deg(p(x) + q(x)) = -\infty < \max\{k, m\}$. ◯

2.2. Polynomial Equations

An equation of the type $ax + b = 0$, where $a, b \in \mathbb{F}$ and $a \ne 0$, is called a linear polynomial equation, or simply a linear equation, and has unique solution

$$x = -\frac{b}{a}.$$

Occasionally a given linear equation may look a little more complicated. However, with the help of some simple algebra, one can reduce the given equation to the form given above.

An equation of the type

$$ax^2 + bx + c = 0,$$

where $a, b, c \in \mathbb{F}$ are constants and $a \ne 0$, is called a quadratic polynomial equation, or simply a quadratic equation. To solve such an equation, we observe first of all that

$$\begin{aligned} ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &= a\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right) \\ &= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right) = 0 \end{aligned}$$

precisely when

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \tag{4}$$

Suppose first of all that $\mathbb{F} = \mathbb{R}$, so that we are considering quadratic equations with real coefficients. Then there are three cases:

(1) If $b^2 - 4ac > 0$, then (4) becomes

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}, \qquad \text{so that} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We therefore have two distinct real solutions for the quadratic equation.

(2) If $b^2 - 4ac = 0$, then (4) becomes

$$\left(x + \frac{b}{2a}\right)^2 = 0, \qquad \text{so that} \qquad x = -\frac{b}{2a}.$$

Indeed, this solution occurs twice, as we shall see later.

(3) If $b^2 - 4ac < 0$, then the right hand side of (4) is negative. It follows that (4) is never satisfied for any real number x, so that the quadratic equation has no real solution.

Suppose next that $\mathbb{F} = \mathbb{C}$, so that we are considering quadratic equations with complex coefficients. Then there are two cases:

(1) If $b^2 - 4ac \neq 0$, then (4) becomes

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}, \qquad \text{so that} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We therefore have two distinct complex solutions for the quadratic equation.

(2) If $b^2 - 4ac = 0$, then (4) becomes

$$\left(x + \frac{b}{2a}\right)^2 = 0, \qquad \text{so that} \qquad x = -\frac{b}{2a}.$$

Again, this solution occurs twice, as we shall see later.

For polynomial equations of degree greater than 2, we do not have general formulae for their solutions. However, we may occasionally be able to find some solutions by inspection. These may help us find other solutions. However, we need to understand better how the solutions of polynomial equations are related to factorization of polynomials. We therefore first study the general problem of attempting to divide a polynomial by another polynomial.

2.3. Division of Polynomials

Let us consider the question of division in $\mathbb{Z}$. It is possible to divide an integer b by a non-zero integer a to get a main term q and remainder r, where $0 \le r < |a|$. In other words, we can find $q, r \in \mathbb{Z}$ such that $b = aq + r$ and $0 \le r < |a|$. In fact, q and r are uniquely determined by a and b. Note that what governs the remainder r is the restriction $0 \le r < |a|$; in other words, the "size" of r.

If one is to propose a theory of division in $\mathbb{F}[x]$, then one needs to find some way to measure the "size" of polynomials. This role is played by the degree. Let us now see what we can do.

EXAMPLE 2.3.1. Let us attempt to divide the polynomial $b(x) = x^4 + 3x^3 + 2x^2 - 4x + 4$ by the non-zero polynomial $a(x) = x^2 + 2x + 2$. Then we can perform long division is a way similar to long division for integers.

$$
\begin{array}{r|l}
 & x^2 + x - 2 \\ \hline
x^2 + 2x + 2 & x^4 + 3x^3 + 2x^2 - 4x + 4 \\
 & x^4 + 2x^3 + 2x^2 \\ \hline
 & x^3 - 4x + 4 \\
 & x^3 + 2x^2 + 2x \\ \hline
 & -2x^2 - 6x + 4 \\
 & -2x^2 - 4x - 4 \\ \hline
 & -2x + 8
\end{array}
$$

Let us explain this a bit more carefully. We attempt to divide the polynomial $x^4 + 3x^3 + 2x^2 - 4x + 4$ by the polynomial $x^2 + 2x + 2$. A factor of x^2 will lift the term x^2 in the "smaller" polynomial to the term x^4 in the "bigger" polynomial, so let us take this first step, and examine the consequences.

$$
\begin{array}{r|l}
 & x^2 \\ \hline
x^2 + 2x + 2 & x^4 + 3x^3 + 2x^2 - 4x + 4 \\
 & x^4 + 2x^3 + 2x^2 \\ \hline
 & x^3 - 4x + 4
\end{array}
$$

We next attempt to divide the polynomial $x^3 - 4x + 4$ by the polynomial $x^2 + 2x + 2$. A factor of x will lift the term x^2 in the "smaller" polynomial to the term x^3 in the "bigger" polynomial, so let us take this second step, and examine the consequences.

$$
\begin{array}{r|l}
 & x^2 + x \\ \hline
x^2 + 2x + 2 & x^4 + 3x^3 + 2x^2 - 4x + 4 \\
 & x^4 + 2x^3 + 2x^2 \\ \hline
 & x^3 - 4x + 4 \\
 & x^3 + 2x^2 + 2x \\ \hline
 & -2x^2 - 6x + 4
\end{array}
$$

We then attempt to divide the polynomial $-2x^2 - 6x + 4$ by the polynomial $x^2 + 2x + 2$. A factor of -2 will reconcile the x^2 terms, so let us take this third step, and complete our task. If we now write $q(x) = x^2 + x - 2$ and $r(x) = -2x + 8$, then $b(x) = a(x)q(x) + r(x)$. Note that $\deg r(x) < \deg a(x)$. We can therefore think of $q(x)$ as the main term and $r(x)$ as the remainder. If we think of the degree as a measure of size, then the remainder $r(x)$ is clearly "smaller" than $a(x)$.

In general, we have the following important result.

PROPOSITION 2B. *Suppose that $a(x), b(x) \in \mathbb{F}[x]$, and that $a(x) \neq 0$. Then there exist unique polynomials $q(x), r(x) \in \mathbb{F}[x]$ such that*
(a) $b(x) = a(x)q(x) + r(x)$; and
(b) either $r(x) = 0$ or $\deg r(x) < \deg a(x)$.

PROOF. Consider all polynomials of the form $b(x) - a(x)Q(x)$, where $Q(x) \in \mathbb{F}[x]$. If there exists $q(x) \in \mathbb{F}[x]$ such that $b(x) - a(x)q(x) = 0$, our proof is complete. Suppose now that $b(x) - a(x)Q(x) \neq 0$ for any $Q(x) \in \mathbb{F}[x]$. Then among all polynomials of the form $b(x) - a(x)Q(x)$, where $Q(x) \in \mathbb{F}[x]$, there must be one with smallest degree. More precisely, $m = \min\{\deg(b(x) - a(x)Q(x)) : Q(x) \in \mathbb{F}[x]\}$ exists. Let $q(x) \in \mathbb{F}[x]$ satisfy $\deg(b(x) - a(x)q(x)) = m$, and let $r(x) = b(x) - a(x)q(x)$. Then $\deg r(x) < \deg a(x)$, for otherwise, writing $a(x) = a_n x^n + \ldots + a_1 x + a_0$ and $r(x) = r_m x^m + \ldots + r_1 x + r_0$, where $m \geq n$, we have

$$r(x) - (r_m a_n^{-1} x^{m-n}) a(x) = b(x) - a(x)\left(q(x) + r_m a_n^{-1} x^{m-n}\right) \in \mathbb{F}[x].$$