Solving Polynomial Equations

3.3

Introduction

Linear and quadratic equations, dealt within Sections 3.1 and 3.2, are members of a class of equations, called **polynomial equations**. These have the general form:

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$$

in which x is a variable and $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are given constants. Also n must be a positive integer and $a_n \neq 0$. Examples include $x^3 + 7x^2 + 3x - 2 = 0$, $5x^4 - 7x^2 = 0$ and $-x^6 + x^5 - x^4 = 0$. In this Section you will learn how to factorise some polynomial expressions and solve some polynomial equations.

Prerequisites

Before starting this Section you should ...

- be able to solve linear and quadratic equations

Learning Outcomes

On completion you should be able to ...

- recognise and solve some polynomial equations

1. Multiplying polynomials together

Key Point 7

A **polynomial expression** is one of the form

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where a_0, a_1, ..., a_n are known coefficients (numbers), $a_n \neq 0$, and x is a variable.

n must be a positive integer.

For example $x^3 - 17x^2 + 54x - 8$ is a polynomial expression in x. The polynomial may be expressed in terms of a variable other than x. So, the following are also polynomial expressions:

$$t^3 - t^2 + t - 3 \qquad z^5 - 1 \qquad w^4 + 10w^2 - 12 \qquad s + 1$$

Note that only non-negative whole number powers of the variable (usually x) are allowed in a polynomial expression. In this Section you will learn how to factorise simple polynomial expressions and how to solve some polynomial equations. You will also learn the technique of **equating coefficients**. This process is very important when we need to perform calculations involving partial fractions which will be considered in Section 6.

The **degree** of a polynomial is the highest power to which the variable is raised. Thus $x^3 + 6x + 2$ has degree 3, $t^6 - 6t^4 + 2t$ has degree 6, and $5x + 2$ has degree 1.

Let us consider what happens when two polynomials are multiplied together. For example

$$(x + 1)(3x - 2)$$

is the product of two first degree polynomials. Expanding the brackets we obtain

$$(x + 1)(3x - 2) = 3x^2 + x - 2$$

which is a second degree polynomial.

In general we can regard a second degree polynomial, or quadratic, as the product of two first degree polynomials, provided that the quadratic can be factorised. Similarly

$$(x - 1)(x^2 + 3x - 7) = x^3 + 2x^2 - 10x + 7$$

is a third degree, or **cubic**, polynomial which is thus the product of a linear polynomial and a quadratic polynomial.

In general we can regard a cubic polynomial as the product of a linear polynomial and a quadratic polynomial or the product of three linear polynomials. This fact will be important in the following Section when we come to factorise cubics.

Key Point 8

A cubic expression can always be formulated as a linear expression times a quadratic expression.

Task

If $x^3 - 17x^2 + 54x - 8 = (x - 4) \times$ (a polynomial), state the degree of the undefined polynomial.

Your solution

Answer

second.

Task

(a) If $3x^2 + 13x + 4 = (x + 4) \times$ (a polynomial), state the degree of the undefined polynomial.

(b) What is the coefficient of x in this unknown polynomial ?

Your solution

(a) (b)

Answer

(a) First. (b) It must be 3 in order to generate the term $3x^2$ when the brackets are removed.

Task

If $2x^2 + 5x + 2 = (x + 2)\times$ (a polynomial), what must be the coefficient of x in this unknown polynomial ?

Your solution

Answer

It must be 2 in order to generate the term $2x^2$ when the brackets are removed.

Two quadratic polynomials are multiplied together. What is the degree of the resulting polynomial?

Your solution

Answer

Fourth degree.

2. Factorising polynomials and equating coefficients

We will consider how we might find the solution to some simple polynomial equations. An important part of this process is being able to express a complicated polynomial into a product of simpler polynomials. This involves **factorisation**.

Factorisation of polynomial expressions can be achieved more easily if one or more of the factors is already known. This requires a knowledge of the technique of 'equating coefficients' which is illustrated in the following example.

Example 23

Factorise the expression $x^3 - 17x^2 + 54x - 8$ given that one of the factors is $(x - 4)$.

Solution

Given that $x - 4$ is a factor we can write

$$x^3 - 17x^2 + 54x - 8 = (x - 4) \times \text{(a quadratic polynomial)}$$

The polynomial must be quadratic because the expression on the left is cubic and $x - 4$ is linear. Suppose we write this quadratic as $ax^2 + bx + c$ where a, b and c are unknown numbers which we need to find. Then

$$x^3 - 17x^2 + 54x - 8 = (x - 4)(ax^2 + bx + c)$$

Removing the brackets on the right and collecting like terms together we have

$$x^3 - 17x^2 + 54x - 8 = ax^3 + (b - 4a)x^2 + (c - 4b)x - 4c$$

Solution (contd.)

Like terms are those which involve the same power of the variable (x).

Equating coefficients means that we compare the coefficients of each term on the left with the corresponding term on the right. Thus if we look at the x^3 terms on each side we see that $x^3 = ax^3$ which implies a must equal 1. Similarly by equating coefficients of x^2 we find $-17 = b - 4a$ With $a = 1$ we have $-17 = b - 4$ so b must equal -13. Finally, equating constant terms we find $-8 = -4c$ so that $c = 2$.

As a check we look at the coefficient of x to ensure it is the same on both sides. Now that we know $a = 1, b = -13, c = 2$ we can write the polynomial expression as

$$x^3 - 17x^2 + 54x - 8 = (x - 4)(x^2 - 13x + 2)$$

Exercises

Factorise into a quadratic and linear product the given polynomial expressions

1. $x^3 - 6x^2 + 11x - 6$, given that $x - 1$ is a factor
2. $x^3 - 7x - 6$, given that $x + 2$ is a factor
3. $2x^3 + 7x^2 + 7x + 2$, given that $x + 1$ is a factor
4. $3x^3 + 7x^2 - 22x - 8$, given that $x + 4$ is a factor

Answers

1. $(x - 1)(x^2 - 5x + 6)$, 2. $(x + 2)(x^2 - 2x - 3)$, 3. $(x + 1)(2x^2 + 5x + 2)$,
4. $(x + 4)(3x^2 - 5x - 2)$.

3. Polynomial equations

When a polynomial expression is equated to zero, a polynomial equation is obtained. Linear and quadratic equations, which you have already met, are particular types of polynomial equation.

Key Point 9

A **polynomial equation** has the form

$$a_nx^n + a_{n-1}x^{n-1} + \ldots a_2x^2 + a_1x + a_0 = 0$$

where a_0, a_1, $\ldots$, a_n are known coefficients, $a_n \neq 0$, and x represents an unknown whose value(s) are to be found.