Solving Systems of Polynomial Equations

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ABSTRACT. One of the most classical problems of mathematics is to solve systems of polynomial equations in several unknowns. Today, polynomial models are ubiquitous and widely applied across the sciences. They arise in robotics, coding theory, optimization, mathematical biology, computer vision, game theory, statistics, machine learning, control theory, and numerous other areas. The set of solutions to a system of polynomial equations is an algebraic variety, the basic object of algebraic geometry. The algorithmic study of algebraic varieties is the central theme of computational algebraic geometry. Exciting recent developments in symbolic algebra and numerical software for geometric calculations have revolutionized the field, making formerly inaccessible problems tractable, and providing fertile ground for experimentation and conjecture.

The first half of this book furnishes an introduction and represents a snapshot of the state of the art regarding systems of polynomial equations. Afficionados of the well-known text books by Cox, Little, and O'Shea will find familiar themes in the first five chapters: polynomials in one variable, Gröbner bases of zero-dimensional ideals, Newton polytopes and Bernstein's Theorem, multidimensional resultants, and primary decomposition.

The second half of this book explores polynomial equations from a variety of novel and perhaps unexpected angles. Interdisciplinary connections are introduced, highlights of current research are discussed, and the author's hopes for future algorithms are outlined. The topics in these chapters include computation of Nash equilibria in game theory, semidefinite programming and the real Nullstellensatz, the algebraic geometry of statistical models, the piecewise-linear geometry of valuations and amoebas, and the Ehrenpreis-Palamodov theorem on linear partial differential equations with constant coefficients.

Throughout the text, there are many hands-on examples and exercises, including short but complete sessions in the software systems `maple`, `matlab`, `Macaulay 2`, `Singular`, `PHC`, and `SOStools`. These examples will be particularly useful for readers with zero background in algebraic geometry or commutative algebra. Within minutes, anyone can learn how to type in polynomial equations and actually see some meaningful results on the computer screen.

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Preface

This book grew out of the notes for ten lectures given by the author at the CBMS Conference at Texas A & M University, College Station, during the week of May 20-24, 2002. Paulo Lima Filho, J. Maurice Rojas and Hal Schenck did a fantastic job of organizing this conference and taking care of more than 80 participants, many of them graduate students working in a wide range of mathematical fields. We were fortunate to be able to listen to the excellent invited lectures delivered by the following twelve leading experts: Saugata Basu, Eduardo Cattani, Karin Gatermann, Craig Huneke, Tien-Yien Li, Gregorio Malajovich, Pablo Parrilo*, Maurice Rojas, Frank Sottile, Mike Stillman*, Thorsten Theobald, and Jan Verschelde*.

Systems of polynomial equations are for everyone: from graduate students in computer science, engineering, or economics to experts in algebraic geometry. This book aims to provide a bridge between mathematical levels and to expose as many facets of the subject as possible. It covers a wide spectrum of mathematical techniques and algorithms, both symbolic and numerical. There are two chapters on applications. The one about statistics is motivated by the author's current research interests, and the one about economics (Nash equilibria) recognizes Dave Bayer's role in the making of the movie *A Beautiful Mind*. (Many thanks, Dave, for introducing me to the stars at their kick-off party in NYC on March 16, 2001).

At the end of each chapter there are about ten exercises. These exercises vary greatly in their difficulty. Some are straightforward applications of material presented in the text while other "exercises" are quite hard and ought to be renamed "suggested research directions". The reader may decide for herself which is which.

We had an inspiring software session at the CBMS conference, and the joy of computing is reflected in this book as well. Sprinkled throughout the text, the reader finds short computer sessions involving polynomial equations. These involve the commercial packages `maple` and `matlab` as well as the freely available packages `Singular`, `Macaulay 2`, `PHC`, and `SOStools`. Developers of the last three programs spoke at the CBMS conference. Their names are marked with a star above.

There are many fine computer programs for solving polynomial systems other than the ones listed above. Sadly, I did not have time to discuss them all. One such program is `CoCoA` which is comparable to `Singular` and `Macaulay 2`. The text book by Kreuzer and Robbiano [**KR00**] does a wonderful job introducing the basics of Computational Commutative Algebra together with examples in `CoCoA`.

Software is necessarily ephemeral. While the mathematics of solving polynomial systems continues to live for centuries, the computer code presented in this book will become obsolete much sooner. I tested it all in May 2002, and it worked well at that time, even on our departmental computer system at UC Berkeley. And if you would like to find out more, each of these programs has excellent documentation.