Limits of functions

mc-TY-limits-2009-1

In this unit, we explain what it means for a function to tend to infinity, to minus infinity, or to a real limit, as x tends to infinity or to minus infinity. We also explain what it means for a function to tend to a real limit as x tends to a given real number. In each case, we give an example of a function that does not tend to a limit at all.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- decide whether a function tends to plus or minus infinity, or to a real limit, as x tends to infinity;
- decide whether a function tends to plus or minus infinity, or to a real limit, as x tends to minus infinity;
- decide whether a function tends to a real limit as x tends to a given real number.

Contents

1. The limit of a function as x tends to infinity

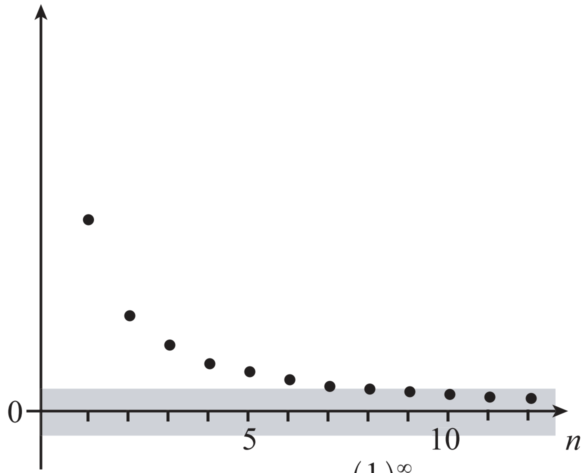
If we have a sequence $(y_n)_{n=1}^{\infty}$, we can say what it means for the sequence to have a limit as n tends to infinity. We write

$$y_n \to l \quad \text{as} \quad n \to \infty$$

if, however small a distance we choose, y_n eventually gets closer to l than that distance, and stays closer. We can also write

$$\lim_{x\to\infty} f(x) = l\,.$$

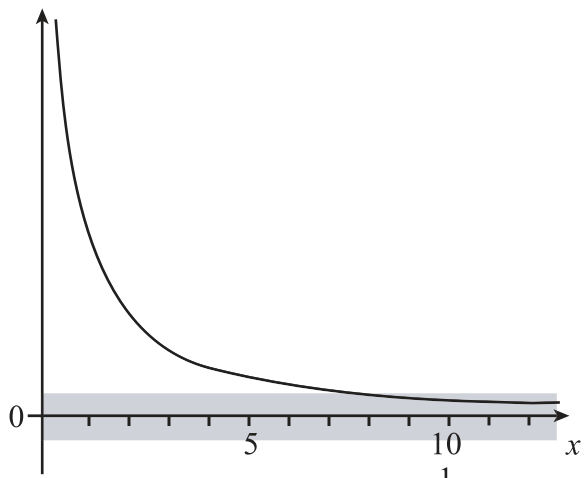
For example, consider the sequence where $y_n = 1/n$. The numbers in this sequence get closer and closer to zero. Whatever positive number we choose, y_n will eventually become smaller than that number, and stay smaller. So y_n eventually gets closer to zero than any distance we choose, and stays closer. We say that the sequence has limit zero as n tends to infinity.



the sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$

We define the limit of a function in a similar way. For example, the points of the sequence $(1/n)_{n=1}^{\infty}$ are also points on the graph of the function $f(x) = 1/x$ for $x > 0$. As x gets larger, $f(x)$ gets closer and closer to zero. In fact, $f(x)$ will get closer to zero than any distance we choose, and will stay closer. We say that $f(x)$ has limit zero as x tends to infinity, and we write

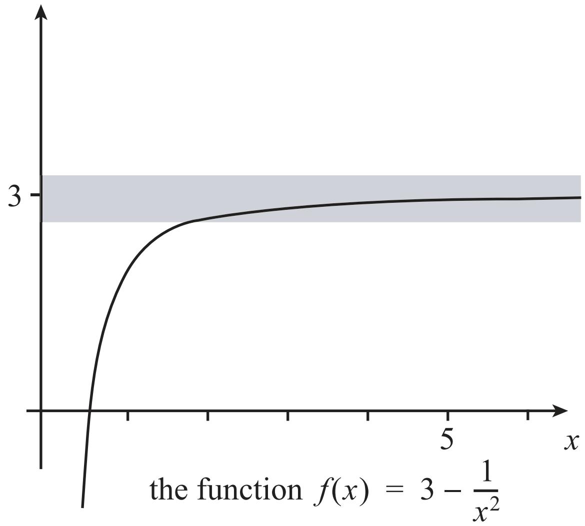
$$f(x) \to 0 \text{ as } x \to \infty\,, \quad \text{or} \quad \lim_{x\to\infty} f(x) = 0\,.$$



the function $f(x) = \frac{1}{x}$

Another example of a function that has a limit as x tends to infinity is the function $f(x) = 3 - 1/x^2$ for $x > 0$. As x gets larger, $f(x)$ gets closer and closer to 3. For any small distance, $f(x)$ eventually gets closer to 3 than that distance, and stays closer. So we say that $f(x)$ has limit 3 as x tends to infinity, and we write

$$f(x) \to 3 \text{ as } x \to \infty\,, \quad \text{or} \quad \lim_{x\to\infty} f(x) = 3\,.$$

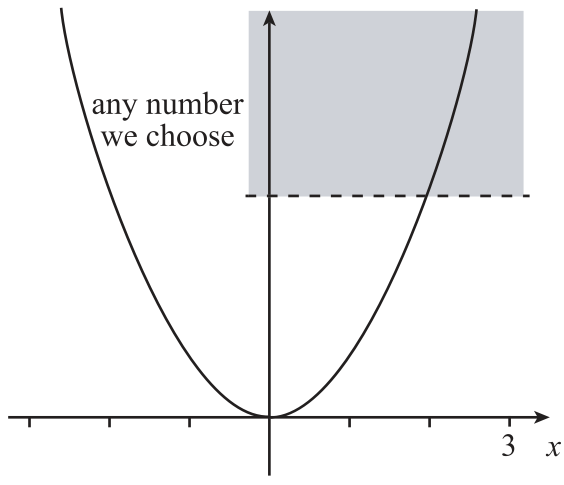


the function $f(x) = 3 - \dfrac{1}{x^2}$

In general, we say that $f(x)$ tends to a real limit l as x tends to infinity if, however small a distance we choose, $f(x)$ gets closer than that distance to l and stays closer as x increases.

Of course, not all functions have real limits as x tends to infinity. Let us look at some other types of behaviour. If we take the function $f(x) = x^2$, we see that $f(x)$ does not get closer to any particular number as x increases. Instead, $f(x)$ just gets larger and larger. At some point, $f(x)$ will get larger than any number we choose, and will stay larger. In this case, we say that $f(x)$ tends to infinity as x tends to infinity, and we write

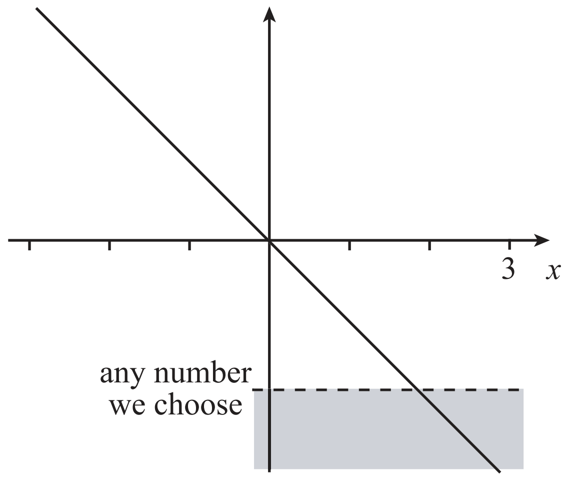
$$f(x) \to \infty \text{ as } x \to \infty\,, \quad \text{or} \quad \lim_{x\to\infty} f(x) = \infty\,.$$



the function $f(x) = x^2$

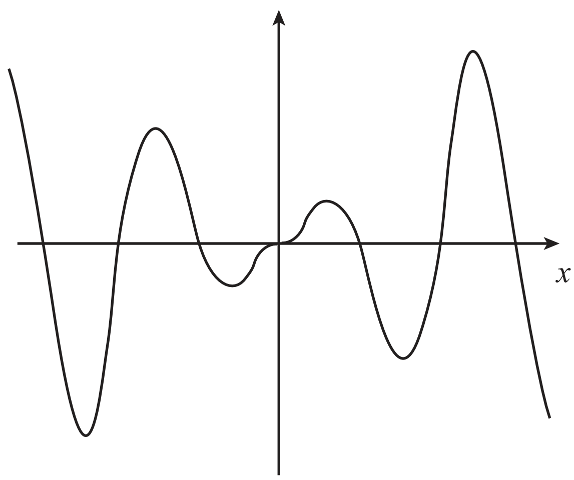
The function $f(x) = -x$ does not have a real limit as x tends to infinity. As x gets larger, this function eventually gets more negative than any number we can choose, and it will stay more negative. In this case, we say that $f(x)$ tends to minus infinity as x tends to infinity, and we write

$$f(x) \to -\infty \text{ as } x \to \infty\,, \quad \text{or} \quad \lim_{x\to\infty} f(x) = -\infty\,.$$



the function $f(x) = -x$

Some functions do not have any kind of limit as x tends to infinity. For example, consider the function $f(x) = x\sin x$. This function does not get close to any particular real number as x gets large, because we can always choose a value of x to make $f(x)$ larger than any number we choose. However $f(x)$ does not tend to infinity, because it does not stay larger than the number we have chosen, but instead returns to zero. For a similar reason, $f(x)$ does not tend to minus infinity. So we cannot talk about the limit of this function as x tends to infinity.



the function $f(x) = x\sin x$

Key Point

The function $f(x)$ has a real limit l as x tends to infinity if, however small a distance we choose, $f(x)$ gets closer than this distance to l and stays closer, no matter how large x becomes.

The function $f(x)$ tends to infinity as x tends to infinity if, however large a number we choose, $f(x)$ gets larger than this number and stays larger, no matter how large x becomes.

The function $f(x)$ tends to minus infinity as x tends to infinity if, however large and negative a number we choose, $f(x)$ gets more negative than this number and stays more negative, no matter how large x becomes.

Exercise 1

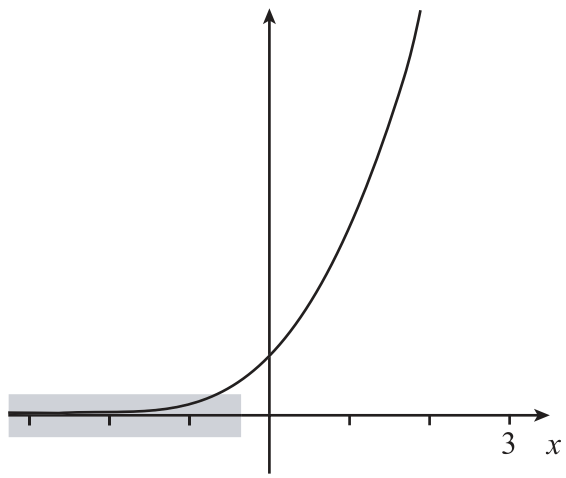
For each of the following functions $f(x)$, find the real limit as $x \to \infty$ if it exists. If it does not exist, state whether the function tends to infinity, tends to minus infinity, or has no limit at all.

(a) $f(x) = 2x^2 - 3x^3$ (b) $f(x) = \tan x$ (c) $f(x) = \dfrac{x+1}{x-1}$ (d) $f(x) = e^{-x} \sin x$
(e) $f(x) = e^x \cos^2 x$ (f) $f(x) = \tan^{-1} x$

2. The limit of a function as x tends to minus infinity

As well as defining the limit of a function as x tends to infinity, we can also define the limit as x tends to minus infinity. Consider the function $f(x) = e^x$. As x becomes more and more negative, $f(x)$ gets closer and closer to zero. However small a distance we choose, $f(x)$ gets closer than that distance to zero, and it stays closer as x becomes more negative. We say that $f(x)$ has limit zero as x tends to minus infinity, and we write

$$f(x) \to 0 \text{ as } x \to -\infty\,, \quad \text{or} \quad \lim_{x \to -\infty} f(x) = 0\,.$$



the function $f(x) = e^x$

