Real Analysis

Course Notes
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1 Introduction

We begin by discussing the motivation for real analysis, and especially for the reconsideration of the notion of integral and the invention of Lebesgue integration, which goes beyond the Riemannian integral familiar from classical calculus.

1. *Usefulness of analysis.* As one of the oldest branches of mathematics, and one that includes calculus, analysis is hardly in need of justification. But just in case, we remark that its uses include:

1. The description of physical systems, such as planetary motion, by dynamical systems (ordinary differential equations);

2. The theory of partial differential equations, such as those describing heat flow or quantum particles;

3. Harmonic analysis on Lie groups, of which $\mathbb{R}$ is a simple example;

4. Representation theory;

5. The description of optimal structures, from minimal surfaces to economic equilibria;

6. The foundations of probability theory;

7. Automorphic forms and analytic number theory; and

8. Dynamics and ergodic theory.

2. *Completeness.* We now motivate the need for a sophisticated theory of measure and integration, called the Lebesgue theory, which will form the first topic in this course.

In analysis it is necessary to take limits; thus one is naturally led to the construction of the real numbers, a system of numbers containing the rationals and closed under limits. When one considers functions it is again natural to work with spaces that are closed under suitable limits. For example, consider the space of continuous functions $C[0,1]$. We might measure the size of a function here by

$$\|f\|_1 = \int_0^1 |f(x)|\, dx.$$

(There is no problem defining the integral, say using Riemann sums).

But we quickly see that there are Cauchy sequences of continuous functions whose limit, in this norm, are discontinuous. So we should extend $C[0,1]$ to a space that is closed under limits. It is not at first even evident that the limiting objects should be *functions.* And if we try to include *all* functions, we are faced with the difficult problem of integrating a general function.

The modern solution to this natural issue is to introduce the idea of *measurable functions*, i.e. a space of functions that is closed under limits and tame enough to integrate. The Riemann integral turns out to be inadequate for these purposes, so a new notion of integration must be invented. In fact we must first examine carefully the idea of the mass or *measure* of a subset $A \subset \mathbb{R}$, which can be though of as the integral of its indicator function $\chi_A(x) = 1$ if $x \in A$ and $= 0$ if $x \notin A$.

3. *Fourier series.* More classical motivation for the Lebesgue integral come from Fourier series.

Suppose $f : [0, \pi] \to \mathbb{R}$ is a reasonable function. We define the Fourier coefficients of f by

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx)\, dx.$$

Here the factor of $2/\pi$ is chosen so that

$$\frac{2}{\pi}\int_0^\pi \sin(nx)\sin(mx)\,dx = \delta_{nm}.$$

We observe that if

$$f(x) = \sum_1^\infty b_n \sin(nx),$$

then at least formally $a_n = b_n$ (this is true, for example, for a finite sum).

This representation of $f(x)$ as a superposition of sines is very useful for applications. For example, $f(x)$ can be thought of as a sound wave, where a_n measures the strength of the frequency n.

Now what coefficients a_n can occur? The orthogonality relation implies that

$$\frac{2}{\pi}\int_0^\pi |f(x)|^2\,dx = \sum_{-\infty}^{\infty} |a_n|^2.$$

This makes it natural to ask if, conversely, for any a_n such that $\sum |a_n|^2 < \infty$, there exists a function f with these Fourier coefficients. The natural function to try is $f(x) = \sum a_n \sin(nx)$.

But why should this sum even exist? The functions $\sin(nx)$ are only bounded by one, and $\sum |a_n|^2 < \infty$ is much weaker than $\sum |a_n| < \infty$.

One of the original motivations for the theory of Lebesgue measure and integration was to refine the notion of function so that this sum really does exist. The resulting function $f(x)$ however need to be Riemann integrable! To get a reasonable theory that includes such Fourier series, Cantor, Dedekind, Fourier, Lebesgue, etc. were led inexorably to a re-examination of the foundations of real analysis and of mathematics itself. The theory that emerged will be the subject of this course.

Here are a few additional points about this example.

First, we could try to define the required space of functions — called $L^2[0,\pi]$ — to simply be the metric completion of, say $C[0,\pi]$ with respect to $d(f,g) = \int |f-g|^2$. The reals are defined from the rationals in a similar fashion. But the question would still remain, can the limiting objects be thought of as functions?

Second, the set of point $E \subset \mathbb{R}$ where $\sum a_n \sin(nx)$ actually converges is liable to be a very complicated set — not closed or open, or even a countable union or intersection of sets of this form. Thus to even begin, we must have a good understanding of subsets of $\mathbb{R}$.

Finally, even if the limiting function $f(x)$ exists, it will generally not be Riemann integrable. Thus we must broaden our theory of integration to deal with such functions. It turns out this is related to the second point — we must again find a good notion for the length or *measure* $m(E)$ of a fairly general subset $E \subset \mathbb{R}$, since $m(E) = \int \chi_E$.

2 Set Theory and the Real Numbers

The foundations of real analysis are given by set theory, and the notion of cardinality in set theory, as well as the axiom of choice, occur frequently in analysis. Thus we begin with a rapid review of this theory. For more details see, e.g. [Hal]. We then discuss the real numbers from both the axiomatic and constructive point of view. Finally we discuss open sets and Borel sets.

In some sense, real analysis is a pearl formed around the grain of sand provided by paradoxical sets. These paradoxical sets include sets that have no reasonable measure, which we will construct using the axiom of choice.

The axioms of set theory. Here is a brief account of the axioms.

- Axiom I. (Extension) A set is determined by its elements. That is, if $x \in A \implies x \in B$ and vice-versa, then $A = B$.
- Axiom II. (Specification) If A is a set then $\{x \in A : P(x)\}$ is also a set.
- Axiom III. (Pairs) If A and B are sets then so is $\{A, B\}$. From this axiom and $\emptyset = 0$, we can now form $\{0, 0\} = \{0\}$, which we call 1; and we can form $\{0, 1\}$, which we call 2; but we cannot yet form $\{0, 1, 2\}$.
- Axiom IV. (Unions) If A is a set, then $\bigcup A = \{x : \exists B, B \in A \ \& \ x \in B\}$ is also a set. From this axiom and that of pairs we can form $\bigcup\{A, B\} = A \cup B$. Thus we can define $x^+ = x + 1 = x \cup \{x\}$, and form, for example, $7 = \{0, 1, 2, 3, 4, 5, 6\}$.
- Axiom V. (Powers) If A is a set, then $\mathcal{P}(A) = \{B : B \subset A\}$ is also a set.
- Axiom VI. (Infinity) There exists a set A such that $0 \in A$ and $x+1 \in A$ whenever $x \in A$. The smallest such set is unique, and we call it $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
- Axiom VII (The Axiom of Choice): For any set A there is a function $c : \mathcal{P}(A) - \{\emptyset\} \to A$, such that $c(B) \in B$ for all $B \subset A$.

Cardinality. In set theory, the natural numbers $\mathbb{N}$ are defined inductively by $0 = \emptyset$ and $n = \{0, 1, \ldots, n-1\}$. Thus n, as a set, consists of exactly n elements.

We write $|A| = |B|$ to mean there is a bijection between the sets A and B; in other words, these sets have the same *cardinality*. A set A is *finite* if $|A| = n$ for some $n \in \mathbb{N}$; it is *countable* if A is finite or $|A| = |\mathbb{N}|$; otherwise, it is *uncountable*.

A countable set is simply one whose elements can be written down in a (possibly finite) list, $(x_1, x_2, \ldots)$. When $|A| = |\mathbb{N}|$ we say A is *countably infinite*.

Inequalities. It is natural to write $|A| \le |B|$ if there is an injective map $A \hookrightarrow B$. By the Schröder–Bernstein theorem (elementary but nontrivial), we have

$$|A| \le |B| \quad \text{and} \quad |B| \le |A| \implies |A| = |B|.$$

The power set. We let A^B denote the set of all maps $f : B \to A$. The *power set* $\mathcal{P}(A) \cong 2^A$ is the set of all subsets of A. A profound observation, due to Cantor, is that

$$|A| < |\mathcal{P}(A)|$$

for any set A. The proof is easy: if $f : A \to \mathcal{P}(A)$ were a bijection, we could then form the set

$$B = \{x \in A \; : \; x \notin f(x)\},$$

but then B cannot be in the image of f, for if $B = f(x)$, then $x \in B$ iff $x \notin B$.

Russel's paradox. We remark that Cantor's argument is closely related to Russell's paradox: if $E = \{X \; : \; X \notin X\}$, then is $E \in E$? Note that the axioms of set theory do not allow us to form the set E!

Countable sets. It is not hard to show that $\mathbb{N} \times \mathbb{N}$ is countable, and consequently:

A countable union of countable sets is countable.

Thus $\mathbb{Z}, \mathbb{Q}$ and the set of algebraic numbers in $\mathbb{C}$ are all countable sets.

Remark: The Axiom of Choice. Recall this axiom states that for any set A ,there is a map $c : \mathcal{P}(A) - \{\emptyset\} \to A$ such that $c(A) \in A$. This axiom is often useful and indeed necessary in proving very general theorems; for example, if there is a surjective map $f : A \to B$, then there is an injective map $g : B \to A$ (and thus $|B| \le |A|$). (Proof: set $g(b) = c(f^{-1}(b))$.)

Another typical application of the axiom of choice is to show: