Chapter 3

Continuity and Limits of Functions

The concept of continuity is an important first step in the analysis leading to differential and integral calculus. It is also an important analytical tool in its own right, with significant practical applications. Fortunately the main theorems are intuitive, though their proofs can be technically challenging. Nonetheless we prove most of the continuity theorems we state, while the remaining theorems we discuss and apply without proof, since they are intuitive and useful but their proofs require background material from at least a junior level real analysis or topology course.

Related to continuity is the concept of limit, which is vital for calculus since it puts calculus on the same rigorous footing as other mathematical disciplines such as algebra and geometry. In our more modern times it has further conceptual appeal, as it is often only possible to approximate the solution to some problem, even though our method of approximation may be arbitrarily close to the actual solution if given enough computing resources. However, the real value of limits lies in its use in confronting an interesting phenomenon. This is the fact that in mathematics we at times find our analysis (algebraic, geometric or otherwise) breaking down at exactly the value of some variable where we would like to compute something; that is, we are allowed to let that variable "approach" the desired value as closely as we would like but it cannot equal that value according to our classical, pre-calculus mathematics. The relatively modern mathematical tool called "limits" can often break through the analytic barrier at that value, in turn opening us to the extensive and spectacularly useful field we call calculus.

Many examples of continuity and limits in action seem straightforward enough, but without a sufficiently deep understanding it is all too easy for students to fall victim to common errors. For this reason we introduce the rather technical definition of continuity here, and develop a method to prove continuity in cases which may seem obvious. Some powerful continuity theorems follow, as do applications. We then employ limits for cases where continuity is "broken," and in numerous other contexts to make calculus possible, and as an analytical tool in its own right.

Understanding limits and continuity sufficiently to avoid numerous common mistakes requires a care and depth of thought which we attempt to foster in this chapter. After continuity, we use a "forms" approach to limits, those forms themselves being ultimately intuitive but nonetheless requiring students to study them carefully and extensively to achieve satisfactory proficiency.[1]

[1]Unfortunately limit computations can be deceiving in that perhaps the majority of problems one first encounters do not require a deep understanding in order to "guess" their correct answers. However the interesting (and more advanced) cases tend to lie outside of those which are easily guessed, and so computing the correct answers in such cases requires a much deeper understanding. We take the approach here that it is better to heavily analyze the simpler cases, so that the later cases are more easily learned.

3.1 Definition of Continuity at a Point

The function $f(x)$ is continuous at $x = a$ if and only if we can guarantee $f(x)$ to be close to the value $f(a)$ by restricting x to be close to a. To rephrase, we say $f(x)$ is continuous at $x = a$ if, given any positive tolerance $\varepsilon > 0$ we choose for $f(x)$ as an approximation for $f(a)$, we can then find a positive tolerance $\delta > 0$ for x as an approximation for a so that δ-tolerance in x allows at most ε tolerance in $f(x)$. The definition below is very technical, but through reflection and exposure to examples, one eventually sees that this is exactly what is required.

Definition 3.1.1 *The function $f(x)$ is continuous at the point $x = a$ if and only if*[2]

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\left(|x - a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon\right). \tag{3.1}$$

This is sometimes called the **epsilon-delta** (ε-δ) definition of continuity. (Note that all values are assumed real, i.e., $\varepsilon, \delta, x, a, f(x), f(a) \in \mathbb{R}$, and so for instance $(\forall x)$ is short for $(\forall x \in \mathbb{R})$.) Now let us examine the various parts of the definition.

$$|f(x) - f(a)| \quad < \quad \varepsilon \tag{3.2}$$

$$|x - a| \quad < \quad \delta \tag{3.3}$$

$$(\forall x)\,(|x - a| < \delta \quad \longrightarrow \quad |f(x) - f(a)| < \varepsilon) \tag{3.4}$$

$$(\forall \varepsilon > 0) \qquad (\exists \delta > 0) \tag{3.5}$$

(3.2): $f(x)$ will be *within* ε of $f(a)$. In other words, the function at x will be near in value to the value of the function at a. How near? Less than ε distance away.

(3.3): x is within δ of a. (Otherwise the implication holds true vacuously, but that case is useless. What is important is what occurs when $|x - a| < \delta$.)

(3.4): The condition that x be within δ of a forces $f(x)$ to be within ε of $f(a)$. In other words, allowing x to stray by less than δ from a keeps $f(x)$ within ε of $f(a)$. By controlling x by allowing it a tolerance of less than δ, we control $f(x)$ to have a tolerance of less than ε.

(3.5): Whatever positive value of ε we choose, we can find a δ which satisfies (3.4). In particular, *no matter how small* we choose $\varepsilon > 0$, we can find a positive δ so that (3.4) is satisfied.

For a final rephrasing, we have the statement that **we can control the tolerance ε in the output $f(x)$ as much as we would like, so long as $\varepsilon > 0$, by controlling the tolerance δ (which must also be positive[3]) in the input variable x.**

[2]Many texts abbreviate statements like (3.1) as follows:

$$(\forall \varepsilon > 0)(\exists \delta > 0)\left(|x - a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon\right),$$

the idea being that the $\forall x$ is understood when we make an unquantified (in x) statement like $|x - a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon$. For a similar example in English, consider the following two statements, usually deemed equivalent:

All Americans have trouble speaking English.

If X is an American, then X has trouble speaking English.

Most see both as false exactly when we can find one American (counterexample) who has no such trouble. In other words, the second statement is as much a "blanket" statement as the first, and is in fact equivalent to the first. We leave the $\forall x$ in (3.1) for precision and to aid in negating the statement, using rules from Section 1.4.

[3]Notice that $\delta = 0$ would be worthless for several reasons. First, the implication would be vacuously true and all functions would be continuous everywhere, since $|x - a| < 0$ would never be satisfied. Second, when—in reality—do we ever have a tolerance of zero in a measurement? Finally, as we explore the implications of continuity, we will see that having positive δ is central to the spirit of what follows, particularly with regards to limits; $\delta > 0$ allows for some "wiggle room" for x near $x = a$, and this wiggle room is crucial to the concept of continuity.

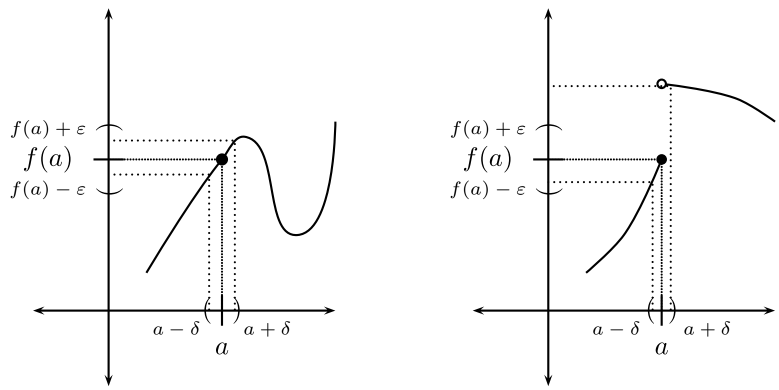

Figure 3.1: The first graph shows a function continuous at $x = a$, illustrating that we can force $f(x)$ to be within any fixed $\varepsilon > 0$ of $f(a)$ by keeping x to within some $\delta > 0$ (depending upon the ε) of a. On the other hand, in the second graph we see an $\varepsilon > 0$ for which no positive δ-tolerance in x can force $f(x)$ to be within ε-tolerance of $f(a)$, and so $f(x)$ is not continuous at $x = a$.

Example 3.1.1 *Show (by a proof!) that the function* $f(x) = 5x - 9$ *is continuous at the point* $x = 2$*, according to the definition* (3.1).

Solution: *First we notice that* $f(2) = 1$*, so we are trying to show the truth of the statement*

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\left(|x-2| < \delta \longrightarrow |f(x) - 1| < \varepsilon\right).$$

Before completing this example, we insert here the following general strategy for all such proofs.

Strategy for Writing ε-δ Proofs

1. Use the statement $|f(x) - f(a)| < \varepsilon$ to see how it can be controlled by $|x - a|$, in particular if $|x - a|$ is a factor of $|f(x) - f(a)|$.

2. If necessary, assume *a priori*[4] that δ is smaller than some fixed positive number to control the other factors contained in $|f(x) - f(a)|$.

3. Find δ as a function of ε, i.e.,
$$\delta = \delta(\varepsilon) \tag{3.6}$$
with $\varepsilon \in (0, \infty)$ (technically, the *domain* of this function $\delta(\varepsilon)$) and $\delta \in (0, \infty)$, and such that the preliminary (or exploratory) analysis indicates that choice of δ satisfies the definition (3.1).

4. Verify that (3.1) holds with this choice of δ, and in doing so write the actual proof.

The first three steps are analysis, or "scratch-work" to determine the form of δ. The final step is the actual proof, though elements of it are often contained in the analysis/scratch-work. Let us apply this strategy to the problem at hand.

[4]Presumptive; before observations. When Step 2 is necessary, we make such suppositions not necessarily based upon observation, but to help focus our search for δ. If continuity is true, (as we will see) we will find that a legitimate δ is still available even with the restriction. In fact, if the limit definition holds for a value of $\delta > 0$, it holds for any smaller positive value δ, so this is not a fatal restriction at all. Note that if $0 < \delta_1 < \delta 2$, and $|x - a| < \delta_1$, then $|x - a| < \delta_2$ as well so we can always take a smaller value for δ in our proof. The upshot is that *a priori* restricting the size of $\delta > 0$ from the start never jeopardizes our ability to prove continuity.

Scratch-work: We want $|f(x) - f(a)| < \varepsilon$ to follow from our choice of δ. We work backwards from that statement, with $f(x) = 5x - 9$, $a = 2$, and $f(a) = f(2) = 1$.

$$\begin{array}{rrcll} & |f(x)-f(a)| & < & \varepsilon & \text{(what we need)} \\ \iff & |f(x)-1| & < & \varepsilon & \\ \iff & |5x-9-1| & < & \varepsilon & \\ \iff & |5x-10| & < & \varepsilon & \\ \iff & 5|x-2| & < & \varepsilon & \\ \iff & |x-2| & < & \frac{1}{5}\varepsilon & \text{(how to get it).} \end{array}$$

We can see from this that, if we take $\delta = \frac{1}{5}\varepsilon$, then $\delta > 0$ (since $\varepsilon > 0$ is assumed), and the bottom could be written $|x - 2| < \delta$. Then the implication could be read from that statement upwards to get $|f(x) - 1| < \varepsilon$. We summarize this in the proof.

<u>Proof</u>: *For any $\varepsilon > 0$ choose $\delta = \frac{1}{5}\varepsilon$. Then $\delta > 0$ exists and satisfies*

$$\begin{aligned} |x-2| < \delta \implies |f(x) - f(2)| &= |(5x-9)-1| \\ &= |5x-10| = \underbrace{5|x-2| < 5\delta}_{\text{since } |x-2|<\delta} = 5 \cdot \frac{1}{5}\varepsilon = \varepsilon, \text{ q.e.d.} \end{aligned}$$

Note that the final line of the proof does imply that $|f(x) - f(2)| < \varepsilon$, with intermediate calculations, some of which one may wish to omit with practice.

Example 3.1.2 *Show that $f(x) = 2x + 3$ is continuous at $x = -5$.*

Scratch-work: Here $a = -5$ and $f(a) = f(-5) = -7$. Hence we wish to find $\delta > 0$ so that

$$|x - (-5)| < \delta \implies |f(x) - (-7)| < \varepsilon,$$

i.e.,

$$|x + 5| < \delta \implies |f(x) + 7| < \varepsilon.$$

Again we work backwards from the conclusion we wish to justify.

$$\begin{array}{rrcl} & |f(x)+7| & < & \varepsilon \\ \iff & |2x+3+7| & < & \varepsilon \\ \iff & |2x+10| & < & \varepsilon \\ \iff & 2|x+5| & < & \varepsilon \\ \iff & |x+5| & < & \frac{1}{2}\varepsilon. \end{array}$$

This time we take $\delta = \frac{1}{2}\varepsilon$, and write the proof.

<u>Proof</u>: *For $\varepsilon > 0$, set $\delta = \frac{1}{2}\varepsilon$. Then $\delta > 0$ exists and satisfies*

$$|x+5| < \delta \implies |f(x)+7| = |2x+3+7| = |2x+10| = 2\underbrace{|x+5|}_{<\delta} < 2\delta = 2\cdot\frac{\varepsilon}{2} = \varepsilon, \text{ q.e.d.}$$

Proving continuity for first-degree polynomials is rather routine at any $x = a$. The strategy is the same for each, with the only complications coming from the signs of the values in question. For completeness we include one more such example.

Example 3.1.3 *Show that $f(x) = 9 - 4x$ is continuous at $x = 2$.*

Scratch-work: Here $a = 2$, $f(a) = f(2) = 1$. Now we must be a little more careful, and will make use of the fact that $|a \cdot b| = |a| \cdot |b|$.

$$\begin{array}{rrcl} & |f(x) - 1| & < & \varepsilon \\ \iff & |9 - 4x - 1| & < & \varepsilon \\ \iff & |-4x + 8| & < & \varepsilon \\ \iff & |(-4)(x - 2)| & < & \varepsilon \\ \iff & 4|x - 2| & < & \varepsilon \\ \iff & |x - 2| & < & \frac{1}{4}\varepsilon. \end{array}$$

Proof: *For $\varepsilon > 0$, choose $\delta = \frac{1}{4}\varepsilon$. Then $\delta > 0$ (exists) and*

$$\begin{aligned} |x - 2| < \delta \implies |f(x) - 1| &= |9 - 4x - 1| = |-4x + 8| \\ &= |(-4)(x - 2)| = 4|x - 2| < 4\delta = 4 \cdot \frac{\varepsilon}{4} = \varepsilon, \text{ q.e.d.} \end{aligned}$$

The function which represents a line is the easiest to confirm continuity at every point. If $f(x) = mx + b$, where $m \neq 0$, it is clear from the geometric meaning of slope m that a variation (absolute value of "rise") of less than ε in height $f(x)$ can be achieved by allowing a variation (absolute value of "run") of less than $\frac{1}{|m|}\varepsilon$ in x. Thus $\delta = \frac{1}{|m|}\varepsilon$ is the largest δ which satisfies the definition of continuity for such a function $f(x)$. (See again our three "linear" examples above, and compare their slopes with our choices of δ.)

Example 3.1.4 *Show that $f(x) = x^2$ is continuous at $x = 0$.*

Scratch-work: Here $a = 0$ and $f(a) = f(0) = 0$. We therefore want to choose $\delta > 0$ such that

$$\begin{aligned} |x - 0| < \delta &\implies |f(x) - 0| < \varepsilon, \quad \text{i.e.,} \\ |x| < \delta &\implies |x|^2 < \varepsilon. \end{aligned}$$

Again we begin with the inequality we would like to result, and see how we might get it.

$$|x|^2 < \varepsilon \iff |x| < \sqrt{\varepsilon}.$$

Here we do have $\iff$ because we are dealing with only positive quantities (recall that $\sqrt{\cdot}$ is an increasing function on $[0, \infty)$). Thus we have a good choice for δ, namely $\delta = \sqrt{\varepsilon}$.

Proof: *For $\varepsilon > 0$, set $\delta = \sqrt{\varepsilon}$. Then $\delta > 0$ exists and satisfies*

$$|x - 0| < \delta \implies |f(x) - f(0)| = |x^2| = |x|^2 < \delta^2 = \left(\sqrt{\varepsilon}\right)^2 = \varepsilon, \quad \text{q.e.d.}$$

It should be clear that we could easily modify this example to show that $f(x) = x^n$ is continuous at $x = 0$ for $n \in \mathbb{N}$. From there it is not hard to show $f(x) = x^{m/n}$ is also continuous at $x = 0$, as long as n is odd and $m, n \in \mathbb{Z}$.[5] Once we stray from $x = 0$, we begin to have more difficulties, as illustrated in the next example.

[5] It is false if n is even, assuming m/n is a reduced fraction. The trouble there is that $x^{m/n} = \left(\sqrt[n]{x}\right)^m$ is undefined for $x < 0$, so the second part of $(|x - 0| < \delta) \longrightarrow (|f(x) - f(0)| < \varepsilon)$ is false for such values of x.