

# DIFFERENTIAL AND INTEGRAL CALCULUS, I

LECTURE NOTES (TEL AVIV UNIVERSITY, FALL 2009)

## CONTENTS

Preliminaries	i
Preparatory reading	i
Reading	i
Problem books	i
Basic notation	ii
Basic Greek letters	iv
1. Real Numbers	1
1.1. Infinite decimal strings	1
1.2. The axioms	1
1.3. Application: solution of equation $s^n = a$	5
1.4. The distance on $\mathbb{R}$	6
2. Upper and lower bounds	8
2.1. Maximum/minimum supremum/infimum	8
2.2. Some corollaries:	10
3. Three basic lemmas:	
Cantor, Heine-Borel, Bolzano-Weierstrass	12
3.1. The nested intervals principle	12
3.2. The finite subcovering principle	13
3.3. The accumulation principle.	13
3.4. Appendix: Countable and uncountable subsets of $\mathbb{R}$	14
4. Sequences and their limits	18
4.1.	18
4.2. Fundamental properties of the limits	19
5. Convergent sequences	22
5.1. Examples	22
5.2. Two theorems	23
5.3. More examples	25
6. Cauchy's sequences. Upper and lower limits.	
Extended convergence	28
6.1. Cauchy's sequences	28
6.2. Upper and lower limits	29
6.3. Convergence in wide sense	31
7. Subsequences and partial limits.	33

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7.1.	Subsequences	33
7.2.	Partial limits	33
8.	Infinite series	36
8.1.		36
8.2.	Examples	36
8.3.	Cauchy's criterion for convergence. Absolute convergence	38
8.4.	Series with positive terms. Convergence tests	38
9.	Rearrangement of the infinite series	42
9.1.	Be careful!	42
9.2.	Rearrangement of the series	42
9.3.	Rearrangement of conditionally convergent series	43
10.	Limits of functions. Basic properties	46
10.1.	Cauchy's definition of limit	46
10.2.	Heine's definition of limit	47
10.3.	Limits and arithmetic operations	48
10.4.	The first remarkable limit: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$	49
10.5.	Limits at infinity and infinite limits	51
10.6.	Limits of monotonic functions	52
11.	The exponential function and the logarithm	53
11.1.	The function $t \mapsto a^t$ .	53
11.2.	The logarithmic function $\log_a x$ .	55
12.	The second remarkable limit. The symbols "o small" and " $\sim$ "	58
12.1.	$\lim_{x \rightarrow \pm\infty} \left(1 + \frac{1}{x}\right)^x = e$	58
12.2.	Infinitesimally small values and the symbols $o$ and $\sim$ .	58
13.	Continuous functions, I	61
13.1.	Continuity	61
13.2.	Points of discontinuity	61
13.3.	Local properties of continuous functions	63
14.	Continuous functions, II	66
14.1.	Global properties of continuous functions	66
14.2.	Uniform continuity	68
14.3.	Inverse functions	70
15.	The derivative	72
15.1.	Definition and some examples	72
15.2.	Some rules	74
15.3.	Derivative of the inverse function and of the composition	75
16.	Applications of the derivative	78
16.1.	Local linear approximation.	78
16.2.	The tangent line	79
16.3.	Lagrange interpolation.	80
17.	Derivatives of higher orders	83
17.1.	Definition and examples	83

17.2.	The Leibniz rule.	86
18.	Basic theorems of the differential calculus:	
	Fermat, Rolle, Lagrange	88
18.1.	Theorems of Fermat and Rolle. Local extrema	88
18.2.	Mean-value theorems	92
19.	Applications of fundamental theorems	96
19.1.	L'Hospital's rule	96
19.2.	Appendix: Algebraic numbers	98
20.	Inequalities	101
20.1.	$\frac{2}{\pi}x \leq \sin x \leq x, \quad 0 \leq x \leq \frac{\pi}{2}$	101
20.2.	$\frac{x}{1+x} < \log(1+x) < x, \quad x > -1, \quad x \neq 0$	101
20.3.	Bernoulli's inequalities	102
20.4.	Young's inequality	103
20.5.	Hölder's inequality	104
20.6.	Minkowski's inequality	105
21.	Convex functions. Jensen's inequality	107
21.1.	Definition	107
21.2.	Fundamental properties of convex functions	109
21.3.		110
21.4.	Jensen's inequality	111
22.	The Taylor expansion	113
22.1.	Local polynomial approximation. Peano's theorem	113
22.2.	The Taylor remainder. Theorems of Lagrange and Cauchy	114
23.	Taylor expansions of elementary functions	117
23.1.	The exponential function	117
23.2.	The sine and cosine functions	118
23.3.	The logarithmic function	119
23.4.	The binomial series	120
23.5.	The Taylor series for $\arctan x$	121
23.6.	Some computations	122
23.7.	Application to the limits	123
24.	The complex numbers	124
24.1.	Basic definitions and arithmetics	124
24.2.	Geometric representation of complex numbers. The argument	125
24.3.	Convergence in $\mathbb{C}$	127
25.	The fundamental theorem of algebra and its corollaries	128
25.1.	The theorem and its proof	128
25.2.	Factoring the polynomials	129
25.3.	Rational functions. Partial fraction decomposition	130
26.	Complex exponential function	133
26.1.	Absolutely convergent series	133
26.2.	The complex exponent	134

## PRELIMINARIES

**Preparatory reading.** These books are intended for high-school students who like math. All three books are great, my personal favorite is the first one.

- (1) R. Courant, H. Robbins, I. Stewart, What is mathematics, Oxford, 1996 (or earlier editions).
- (2) T. W. Korner, The pleasures of counting, Cambridge U. Press, 1996.
- (3) K. M. Ball, Strange curves, counting rabbits, and other mathematical explorations, Princeton University Press, 2003.

**Reading.** There are many good textbooks in analysis, though I am not going to follow any of them too closely. The following list reflects my personal taste:

- (1) V. A. Zorich, Mathematical analysis, vol.1, Springer, 2004.
- (2) A. Browder, Mathematical analysis. An introduction. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1996.
- (3) R. Courant and F. John, Introduction to calculus and analysis, vol.1, Springer, 1989 (or earlier editions).
- (4) D. Maizler, Infinitesimal calculus (in Hebrew).
- (5) G. M. Fihtengol'tz, Course of Differential and Integral Calculus, vol. I (in Russian)
- (6) E. Hairer, G. Wanner, Analysis by its history, Springer, 1996.

The last book gives a very interesting and motivated exposition of the main ideas of this course given in the historical perspective.

You may find helpful informal discussions of various ideas related to this course (as well to the other undergraduate courses) at the web page of Timothy Gowers:

[www.dpmms.cam.ac.uk/~wtg10/mathsindex.html](http://www.dpmms.cam.ac.uk/~wtg10/mathsindex.html)

I suppose that the students attend in parallel with this course the course “Introduction to the set theory”, or the course “Discrete Mathematics”. The notes (in Hebrew) of Moshe Jarden might be useful:

[www.math.tau.ac.il/~jarden/Courses/set.pdf](http://www.math.tau.ac.il/~jarden/Courses/set.pdf)

**Problem books.** For those of you who are interested to try to solve more difficult and interesting problems and exercises, I strongly recommend to look at two excellent collections of problems:

- (1) B. M. Makarov, M. G. Goluzina, A. A. Lodkin, A. N. Podkorytov, Selected problems in real analysis, American Mathematical Society, 1992.
- (2) G. Polya, G. Szegö, Problems and theorems in analysis (2 volumes) Springer, 1972 (there are earlier editions).

**Basic notation.***Symbols from logic.*

$\vee$	or
$\wedge$	and
$\neg$	negation
$\implies$	yields
$\iff$	is equivalent to

*Example:*  $(x^2 - 3x + 2 = 0) \iff ((x = 1) \vee (x = 2))$

*Quantifiers:*

$\exists$	exists
$\exists!$	exists and unique (warning: this notation isn't standard)
$\forall$	for every

*Set-theoretic notation.*

$\in$	belongs
$\notin$	does not belong
$\subset$	subset
$\emptyset$	empty set
$\cap$	intersection of sets
$\cup$	union of sets
$\#(X)$	cardinality of the set $X$
$X \setminus Y = \{x \in X : x \notin Y\}$	complement to $Y$ in $X$

*Example:*  $(X \subset Y) := \forall x ((x \in X) \implies (x \in Y))$

We shall freely operate with these notion during the course. Usually, the sets we deal with are subsets of the set of real numbers  $\mathbb{R}$ .

*Subsets of reals:*

$\mathbb{N}$	natural numbers (positive integers)
$\mathbb{Z}$	integers
$\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$	non-negative integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$	closed interval (one point sets are closed intervals as well)
$(a, b) := \{x \in \mathbb{R} : a < x < b\}$	open interval
$(a, b]$ and $[a, b)$	semi-open intervals

*Sums and products.*

$$\sum_{j=1}^n a_j = a_1 + a_2 + \dots + a_n$$

$$\prod_{j=1}^n a_j = a_1 \cdot a_2 \cdot \dots \cdot a_n$$

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