A Course in Riemannian Geometry

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1 Smooth Manifolds

1.1 Smooth Manifolds

A topological space M is said a *topological manifold* of dimension n if it is metrizable (i.e., there exists a distance function d on M which generates the topology of M) and every point of M has an open neighbourhood homeomorphic to an open set in n-dimensional Euclidean space $\mathbb{R}^n$.

Let M be a topological manifold. A *continuous coordinate system* defined over an open set U in M is defined to be an n-tuple $(x^1, x^2, \ldots, x^n)$ of continuous real-valued functions on U such that the map $\varphi\colon U \to \mathbb{R}^n$ defined by

$$\varphi(u) = \bigl(x^1(u), x^2(u), \ldots, x^n(u)\bigr)$$

maps U homeomorphically onto some open set in $\mathbb{R}^n$. The domain U of the coordinate system $(x^1, x^2, \ldots, x^n)$ is referred to as a *coordinate patch* on M.

Two continuous coordinate systems $(x^1, x^2, \ldots, x^n)$ and $(y^1, y^2, \ldots, y^n)$ defined over coordinate patches U and V are said to be *smoothly compatible* if the coordinates $(x^1, x^2, \ldots, x^n)$ depend smoothly on $(y^1, y^2, \ldots, y^n)$ and vica versa on the overlap $U \cap V$ of the coordinate patches. Note in particular that two coordinate systems are smoothly compatible if the corresponding coordinate patches are disjoint.

A *smooth atlas* on M is a collection of continuous coordinate systems on M such that the following two conditions hold:—

(i) every point of M belongs to the coordinate patch of at least one of these coordinate systems,

(ii) the coordinate systems in the atlas are smoothly compatible with one another.

Let $\mathcal{A}$ be a smooth atlas on a topological manifold M of dimension n. Let $(u^1, u^2, \ldots, u^n)$ and $(v^1, v^2, \ldots, v^n)$ be continuous coordinate systems, defined over coordinate patches U and V respectively. If the coordinate systems (u^i) and (v^i) are smoothly compatible with all the coordinate systems in the atlas $\mathcal{A}$ then they are smoothly compatible with each other. Indeed suppose that $U \cap V \neq 0$, and let m be a point of $U \cap V$. Then (by condition (i) above) there exists a coordinate system (x^i) belonging to the atlas $\mathcal{A}$ whose coordinate patch includes that point m. But the coordinates (v^i) depend smoothly on the coordinates (x^i), and the coordinates (x^i) depend smoothly on the coordinates (u^i) around m (since the coordinate systems (u^i) and (v^i) are smoothly compatible with all coordinate systems in the atlas $\mathcal{A}$). It follows from the Chain Rule that the coordinates (v^i) depend smoothly on the coordinates (u^i) around m, and similarly the coordinates (u^i) depend smoothly on the coordinates (v^i). Therefore the continuous coordinate systems (u^i) and (v^i) are smoothly compatible with each other.

We deduce that, given a smooth atlas $\mathcal{A}$ on a topological manifold M, we can enlarge $\mathcal{A}$ by adding to to $\mathcal{A}$ all continuous coordinate systems on M that are smoothly compatible with each of the coordinate systems of $\mathcal{A}$. In this way we obtain a smooth atlas on M which is *maximal* in the sense that any coordinate system smoothly compatible with all the coordinate systems in the atlas already belongs to the atlas.

Definition A *smooth manifold* $(M, \mathcal{A})$ consists of a topological manifold M together with a maximal smooth atlas $\mathcal{A}$ of coordinate systems on M. A *smooth coordinate system* $(x^1, x^2, \ldots, x^n)$ on M is a coordinate system belonging to the maximal smooth atlas $\mathcal{A}$.

Note that $\mathbb{R}^n$ is a smooth manifold of dimension n. The maximal smooth atlas on $\mathbb{R}^n$ consists of all (curvilinear) coordinate systems that are smoothly compatible with the standard Cartesian coordinate system on $\mathbb{R}^n$.

1.2 Submanifolds

Let M be a subset of a k-dimensional smooth manifold N. We say that M is a smooth *embedded submanifold* of N of dimension n if, given any point m of M, there exists a smooth coordinate system $(u^1, u^2, \ldots, u^k)$ defined over some open set U in N, where $m \in U$, with the property that

$$M \cap U = \{p \in U : u^i(p) = 0 \text{ for } i = n+1, \ldots, k\}.$$

Given such a coordinate system $(u^1, u^2, \ldots, u^k)$, the restrictions of the coordinate functions $u^1, u^2, \ldots, u^n$ to $U \cap M$ provide a coordinate system on M around the point m. The collection of all such coordinate systems constitutes a smooth atlas on M. Thus any smooth embedded submanifold M of a smooth manifold N is itself a smooth manifold (with respect to the unique maximal smooth atlas containing the smooth atlas on M just described).

Example Consider the unit sphere S^n in $\mathbb{R}^{n+1}$ consisting of those vectors $\mathbf{x}$ in $\mathbb{R}^{n+1}$ satisfying $|\mathbf{x}| = 1$. Given any integer i between 1 and $n+1$, let

$$u^j(\mathbf{x}) = \begin{cases} x^j & \text{if } j < i; \\ x^{j+1} & \text{if } i \leq j \leq n; \\ (x^1)^2 + (x^2)^2 + \cdots + (x^{n+1})^2 - 1 & \text{if } j = n+1. \end{cases}$$

Then $(u^1, u^2, \ldots, u^{n+1})$ is a smooth coordinate system on U_i^+ and on U_i^-, where

$$U_i^+ = \{\mathbf{x} \in \mathbb{R}^{n+1} : x^i > 0\}, \qquad U_i^- = \{\mathbf{x} \in \mathbb{R}^{n+1} : x^i < 0\},$$

and

$$S^n \cap U_i^\pm = \{\mathbf{x} \in U_i^\pm : u^{n+1}(\mathbf{x}) = 0\}.$$

Moreover S^n is covered by $U_1^\pm, U_2^\pm, \ldots, U_{n+1}^\pm$. This shows that S^n is a smooth embedded submanifold of $\mathbb{R}^{n+1}$.

1.3 Smooth Mappings between Smooth Manifolds

Let M and N be smooth manifolds of dimension n and k respectively. A mapping $\varphi\colon M \to N$ from M to N is said to be *smooth* around a point m of M if, given smooth coordinate systems $(x^1, x^2, \ldots, x^n)$ and $(y^1, y^2, \ldots, y^k)$ around m and $\varphi(m)$, the coordinates $(y^1(\varphi(u)), y^2(\varphi(u)), \ldots, y^k(\varphi(u))$ of $\varphi(u)$ depend smoothly on the coordinates $(x^1(u), x^2(u), \ldots, x^n(u))$ of u for all points u belonging to some sufficiently small neighbourhood of m. (Note that if there exist smooth coordinate systems $(x^1, x^2, \ldots, x^n)$ and $(y^1, y^2, \ldots, y^k)$ around m and $\varphi(m)$ for which this condition is satisfied, then the condition is satisfied for all such smooth coordinate systems around m and $\varphi(m)$; this follows easily from the fact that a composition of smooth functions is smooth.) The mapping $\varphi\colon M \to N$ is said to be *smooth* if it is smooth around every point of M.

1.4 Bump Functions and Partitions of Unity

Let $f\colon X \to \mathbb{R}$ be a real-valued function defined defined over a topological space X. The *support* $\operatorname{supp} f$ of f is defined to be the closure of the set $\{x \in X : f(x) \neq 0\}$. Thus $\operatorname{supp} f$ is the smallest closed set in X with the property that the function f vanishes on the complement of that set.

Lemma 1.1 *Let U be an open set in a smooth manifold M of dimension n, and let m be a point of M. Then there exists an open subset V of U containing the point m and a smooth non-negative function $f\colon M \to \mathbb{R}$ such that* $\operatorname{supp} f \subset U$ *and $f(v) = 1$ for all $v \in V$.*

Proof We may assume, without loss of generality, that U is contained in the coordinate patch of some smooth coordinate system $(x^1, x^2, \ldots, x^n)$ and that $x^i(m) = 0$ for $i = 1, 2, \ldots, n$. Thus it suffices to show that, given any $r > 0$,