Probability and Geometry on Groups

Lecture notes for a graduate course

Gábor Pete
Technical University of Budapest
http://www.math.bme.hu/~gabor

August 3, 2015
Work in progress. Comments are welcome.

Abstract

These notes have grown (and are still growing) out of two graduate courses I gave at the University of Toronto. The main goal is to give a self-contained introduction to several interrelated topics of current research interests: the connections between

1) coarse geometric properties of Cayley graphs of infinite groups;
2) the algebraic properties of these groups; and
3) the behaviour of probabilistic processes (most importantly, random walks, harmonic functions, and percolation) on these Cayley graphs.

I try to be as little abstract as possible, emphasizing examples rather than presenting theorems in their most general forms. I also try to provide guidance to recent research literature. In particular, there are presently over 150 exercises and many open problems that might be accessible to PhD students. It is also hoped that researchers working either in probability or in geometric group theory will find these notes useful to enter the other field.

Contents

Preface

These notes have grown (and are still growing) out of two graduate courses I gave at the University of Toronto: *Probability and Geometry on Groups* in the Fall of 2009, and *Percolation in the plane, on* $\mathbb{Z}^d$*, and beyond* in the Spring of 2011. I am still adding material and polishing the existing parts, so at the end I expect it to be enough for two semesters, or even more. Large portions of the first drafts were written up by the nine students who took the first course for credit: Eric Hart, Siyu Liu, Kostya Matveev, Jim McGarva, Ben Rifkind, Andrew Stewart, Kyle Thompson, Lluís Vena, and Jeremy Voltz — I am very grateful to them. That first course was completely introductory: some students had not really seen probability before this, and only few had seen geometric group theory. Here is the course description:

> Probability is one of the fastest developing areas of mathematics today, finding new connections to other branches constantly. One example is the rich interplay between large-scale geometric properties of a space and the behaviour of stochastic processes (like random walks and percolation) on the space. The obvious best source of discrete metric spaces are the Cayley graphs of finitely generated groups, especially that their large-scale geometric (and hence, probabilistic) properties reflect the algebraic properties. A famous example is the construction of expander graphs using group representations, another one is Gromov's theorem on the equivalence between a group being almost nilpotent and the polynomial volume growth of its Cayley graphs. The course will contain a large variety of interrelated topics in this area, with an emphasis on open problems.

What I had originally planned to cover turned out to be ridiculously much, so a lot had to be dropped, which is also visible in the present state of these notes. The main topics that are still missing are Gromov-hyperbolic groups and their applications to the construction of interesting groups, metric embeddings of groups in Hilbert spaces, more on the construction and applications of expander graphs, more on critical spatial processes in the plane and their scaling limits, and a more thorough study of Uniform Spanning Forests and ℓ^2-Betti numbers — I am planning to improve the notes regarding these issues soon. Besides research papers I like, my primary sources were [DrK09], [dlHar00] for geometric group theory and [LyPer14], [Per04], [Woe00] for probability. I did not use more of [HooLW06], [Lub94], [Wil09] only because of the time constraints. There are proofs or even sections that follow rather closely one of these books, but there are always differences in the details, and the devil might be in those. Also, since I was a graduate student of Yuval Peres not too long ago, several parts of these notes are strongly influenced by his lectures. In particular, Chapter 9 contains paragraphs that are almost just copied from some unpublished notes of his that I was once editing. There is one more recent book, [Gri10], whose first few chapters have considerable overlap with the more introductory parts of these notes, although I did not look at that book before having finished most of these notes. Anyway, the group theoretical point of view is missing from that book entirely.

With all these books available, what is the point in writing these notes? An obvious reason is that it is rather uncomfortable for the students to go to several different books and start reading them somewhere from their middle. Moreover, these books are usually for a bit more specialized audience, so either nilpotent groups or martingales are not explained carefully. So, I wanted to add my favourite explanations and examples to everything, and include proofs I have not seen elsewhere in the literature. And there was a very important goal I had: presenting the material in constant conversation between the probabilistic and geometric group theoretical ideas. I hope this will help not only students, but also researchers from either field get interested and enter the other territory.

There are presently over 150 exercises, in several categories of difficulty: the ones without any stars should be doable by everyone who follows the notes, though they are often not quite trivial; * means it is a challenge for the reader; ** means that I think I would be able to do it, but it would be a challenge for me; *** means it is an open problem. Part of the grading scheme was to submit exercise solutions worth 8 points, where each exercise was worth $2^{\#\text{ of stars}}$. There are also conjectures and questions in the notes — the difference compared to the *** exercises is that, according to my knowledge or feeling, the *** exercises have not been worked on yet thoroughly enough, so I want to encourage the reader to try and attack them. Of course, this does not necessarily mean that all conjectures are hard, neither that any of the *** exercises are doable. . . . But, e.g., for a PhD topic, I would personally suggest starting with the *** exercises.

Besides my students and the books mentioned above, I am grateful to Ági Backhausz, Alex Bloemendal, Damien Gaboriau, Gady Kozma, Russ Lyons, Sébastien Martineau, Péter Mester, Yuval Peres, Mark Sapir, Andreas Thom, Ádám Timár, László Márton Tóth, Todor Tsankov and Bálint Virág for conversations and comments.