Trigonometric Functions

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Topics Covered:

Reminder: relationship between degrees and radians
The unit circle
Definitions of trigonometric functions for a right triangle
Definitions of trigonometric functions for a unit circle
Exact values for trigonometric functions of most commonly used angles
Trigonometric functions of any angle θ' in terms of angle θ in quadrant I
Trigonometric functions of negative angles
Some useful relationships among trigonometric functions
Double angle formulas
Half angle formulas
Angle addition formulas
Sum, difference and product of trigonometric functions
Graphs of trigonometric functions
Inverse trigonometric functions
Principal values for inverse trigonometric functions
Relations between inverse trigonometric functions
Graphs of inverse trigonometric functions
Using trigonometric functions: components of a vector
Using trigonometric functions: phase shift of a wave
Derivatives of trigonometric functions

Reminder: Relationship Between Degrees and Radians

A radian is defined as an angle θ subtended at the center of a circle for which the arc length is equal to the radius of that circle (see Fig.1).

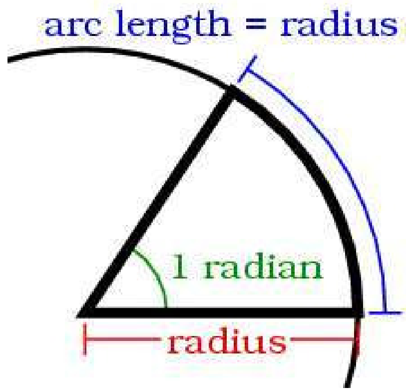

Fig.1. Definition of a radian.

The circumference of the circle is equal to $2\pi R$, where R is the radius of the circle. Consequently, 360°=2π radians. Thus,

$$1 \text{ radian} = 360°/2\pi \approx 57.296°$$

$$\mathbf{1° = (2\pi/360) \text{ radians} \approx 0.01745 \text{ radians}}$$

The Unit Circle

In mathematics, a unit circle is defined as a circle with a radius of 1. Often, especially in applications to trigonometry, the unit circle is centered at the origin (0,0) in the coordinate plane. The equation of the unit circle in the coordinate plane is

$$x^2 + y^2 = 1.$$

As mentioned above, the unit circle is taken to be 360°, or 2π radians. We can divide the coordinate plane, and therefore, the unit circle, into 4 quadrants. The first quadrant is defined in terms of coordinates by $x>0$, $y>0$, or, in terms of angles, by $0°<\theta<90°$, or $0<\theta<\pi/2$. The second quadrant is defined by $x<0$, $y>0$, or $90°<\theta<180°$, or $\pi/2<\theta<\pi$. The third quadrant is defined by $x<0$, $y<0$, or $180°<\theta<270°$, or $\pi<\theta<3\pi/2$. Finally, the fourth quadrant is defined by $x>0$, $y<0$, or $270°<\theta<360°$, or $3\pi/2<\theta<2\pi$.

Trigonometric Functions

Definitions of Trigonometric Functions For a Right Triangle

A right triangle is a triangle with a right angle (90°) (See Fig.2).

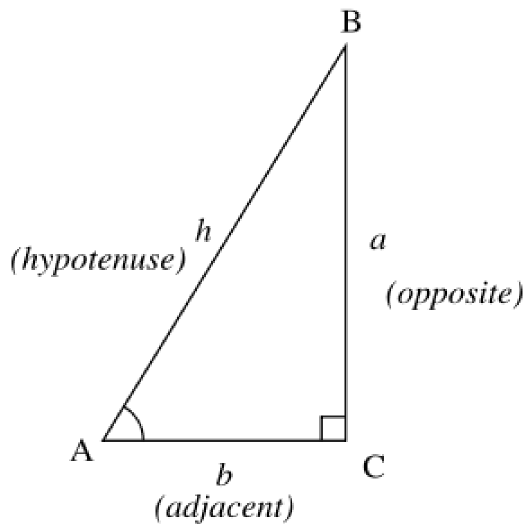

Fig.2. Right triangle.

For every angle θ in the triangle, there is the side of the triangle adjacent to it (from here on denoted as "adj"), the side opposite of it (from here on denoted as "opp"), and the hypotenuse (from here on denoted as "hyp"), which is the longest side of the triangle located opposite of the right angle. For angle θ, the trigonometric functions are defined as follows:

$$\text{sine of } \theta = \sin\theta = \frac{opp}{hyp}$$

$$\text{cosine of } \theta = \cos\theta = \frac{adj}{hyp}$$

$$\text{tangent of } \theta = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{opp}{adj}$$

$$\text{cotangent of } \theta = \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{adj}{opp}$$

$$\text{secant of } \theta = \sec\theta = \frac{1}{\cos\theta} = \frac{hyp}{adj}$$

$$\text{cosecant of } \theta = \csc\theta = \frac{1}{\sin\theta} = \frac{hyp}{opp}$$

Definitions of Trigonometric Functions For a Unit Circle

In the unit circle, one can define the trigonometric functions cosine and sine as follows. If (x,y) is a point on the unit cirlce, and if the ray from the origin (0,0) to that point (x,y) makes an angle θ with the positive x-axis, (such that the counterclockwise direction is considered positive), then,

$$\cos\theta = x/1 = x$$
$$\sin\theta = y/1 = y$$

Then, each point (x,y) on the unit circle can be written as ($\cos\theta$, $\sin\theta$). Combined with the equation $x^2 + y^2 = 1$, the definitions above give the relationship $\sin^2\theta+\cos^2\theta=1$. In addition, other trigonometric functions can be defined in terms of x and y:

$$\tan\theta = \sin\theta/\cos\theta = y/x$$
$$\cot\theta = \cos\theta/\sin\theta = x/y$$
$$\sec\theta = 1/\cos\theta = 1/x$$
$$\csc\theta = 1/\sin\theta = 1/y$$

Fig.3 below shows a unit circle in the coordinate plane, together with some useful values of angle θ, and the points (x,y)=($\cos\theta$, $\sin\theta$), that are most commonly used (also see table in the following section).

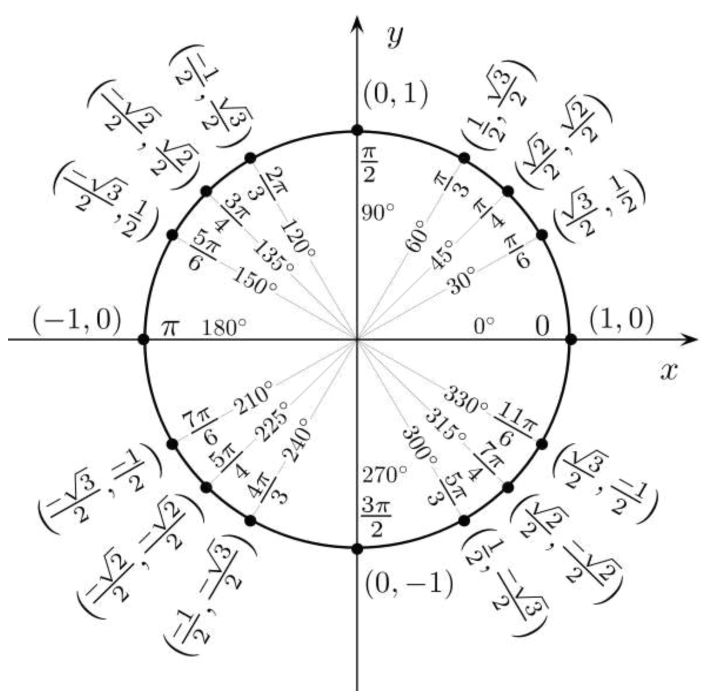

Fig.3. Most commonly used angles and points of the unit circle.

Note: For θ in quadrant I, $\sin\theta>0$, $\cos\theta>0$; for θ in quadrant II, $\sin\theta>0$, $\cos\theta<0$; for θ in quadrant III, $\sin\theta<0$, $\cos\theta<0$; and for θ in quadrant IV, sin θ

Exact Values for Trigonometric Functions of Most Commonly Used Angles

θ in degrees	θ in radians	sinθ	cosθ	tanθ
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	undefined
180	π	0	-1	0
270	$\frac{3\pi}{2}$	-1	0	undefined
360	2π	0	1	0

Note: Exact values for other trigonometric functions (such as cotθ, secθ, and cscθ) as well as trigonometric functions of many other angles can be derived by using the following sections.

Trigonometric Functions of Any Angle θ' in Terms of Angle θ in Quadrant I

θ'	sinθ'	cosθ'	tanθ'	θ'	sinθ'	cosθ'	tanθ'
90°+θ π/2+θ	cosθ	−sinθ	-cotθ	90°−θ π/2-θ	cosθ	sinθ	cotθ
180°+θ π+θ	-sinθ	-cosθ	tanθ	180°−θ π−θ	sinθ	-cosθ	-tanθ
270°+θ 3π/2+θ	-cosθ	sinθ	-cotθ	270°−θ 3π/2-θ	-cosθ	-sinθ	cotθ
k(360°)+θ k(2π)+θ k=integer	sinθ	cosθ	tanθ	k(360°)−θ k(2π)−θ k=integer	-sinθ	cosθ	-tanθ