Notes on Differential Equations

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Preface

These *Notes on Differential Equations* are an introduction and invitation. The focus is on

1. important models
2. calculus (review?) in applied contexts

I may point out that the title is not *Solving* Differential Equations; we derive them, discuss them, review calculus background for them, apply them, sketch and compute them, and also solve them and interpret the solutions. This breadth is new to many students.

The notes, available for many years on my web page, have evolved from lectures I have given while teaching the Engineering Mathematics courses at Cornell University. They could be used for an introductory unified course on ordinary and partial differential equations. There is minimal manipulation and a lot of emphasis on the teaching of concepts by example.

For background on calculus see

- Lax, P., and Terrell, M., *Calculus With Applications*, Springer, 2014.

The focus on key models here was influenced by the Lax Terrell book. In a few places we assume familiarity with the divergence theorem. For further information see:

- Churchill, Ruel V., *Fourier Series and Boundary Value Problems,* McGraw Hill, 1941
- Hubbard, John H., and West, Beverly H., *Differential Equations, a Dynamical Systems Approach, Parts 1 and 2,* Springer, 1995 and 1996.
- and the software discussed in Lecture 5.

Some of the exercises have the format *"What's rong with this?"*. These are either questions asked by students or errors taken from test papers of students in this class, so it could be quite beneficial to study them.

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Contents

1 The Banker's Equation **1**

1.1 Slope Fields 2

2 A Gallery of Differential Equations **5**

3 The Transport Equation **7**

3.1 A Conservation Law 7

3.2 Traveling Waves 8

4 The Logistic Population Model **11**

5 Existence and Uniqueness and Software **14**

6 Newton's Law of Cooling **17**

6.1 Investments 20

7 Exact equations for Air and Steam **21**

8 Euler's Numerical Method **24**

9 Spring-mass oscillations **28**

9.1 Conservation laws and uniqueness 31

10 Applications of Complex Numbers **33**

10.1 Exponential and characteristic equation 33

10.2 The Fundamental Theorem of Algebra 38

10.3 A forced oscillator 39

11 Three masses oscillate **42**

12 Boundary Value Problems **47**

13 The Conduction of Heat **50**
13.1 Walk the line . . . 53

14 Initial Boundary Value Problems for the Heat Equation **57**
14.1 Insulation . . . 58
14.2 Product Solutions . . . 59
14.3 Superposition . . . 61

15 The Wave Equation **65**

16 Application of Power Series: a Drum model **69**
16.1 A new Function for the Drum model, J_0 . . . 73
16.2 But what does the drum Sound like? . . . 75

17 The Euler equation for Fluid Flow, and Acoustic Waves **77**
17.1 Sound . . . 81

18 The Laplace Equation **82**
18.1 Laplace leads to Fourier . . . 85
18.2 Fourier's Dilemma . . . 87
18.3 Fourier answered by Orthogonality . . . 89

19 Application to the weather? **92**

1 The Banker's Equation

TODAY: An example involving your bank account, and nice pictures called slope fields (or direction fields). How to read a differential equation.

Welcome to the world of differential equations! They describe many processes in the world around you, but of course we'll have to convince you of that. Today we are going to give an example, and find out what it means to read a differential equation.

A differential equation is an equation which contains a derivative of an unknown function. It tells something about a rate of change, from which we hope to deduce facts about the function. Here is a differential equation.

$$\frac{dy}{dt} = .01y$$

It might represent your bank account, where the balance is $y(t)$ at a time t years after you open the account, and the account is earning 1% interest. Regardless of the specific interpretation, let's see what the equation says. Since we see the term dy/dt we can tell that y is a function of t, and that the rate of change is a multiple, namely .01, of the value of y itself. We definitely should always write $y(t)$ instead of just y, and we will sometimes, but it is traditional to be sloppy.

For example, if y happens to be 2000 at a particular time t, the rate of change of y is then $.01(2000) = 20$, and the units of this rate are dollars/year. From calculus we know that y is increasing whenever y' is positive, thus whenever y is positive. I hope your bank balance is positive!

PRACTICE: What do you estimate the balance will be, roughly, a year from now, if it is 2000 and is growing at 20 dollars/year?
This is not supposed to be a hard question. By the way, when I ask a question, don't cheat yourself by ignoring it. Think about it, and future things will be easier.

Later when y is, say, 5875.33, its rate of change will be $.01(5875.33) = 58.7533$ which is much faster. We'll sometimes refer to $y' = .01y$ as the banker's equation.

Do you begin to see how you can get useful information from a differential equation fairly easily, by just reading it carefully? One of the most important skills to learn about differential equations is how to read them. For example in the equation

$$y' = .01y - 10$$