Differential Equations

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Abstract

These are notes for an undergraduate course on differential equations; please send corrections, suggestions and notes to courses@suchideas.com The author's homepage for all courses may be found on his website at SuchIdeas.com, which is where updated and corrected versions of these notes can also be found.

Thanks go to Professor G. Worster for allowing me to use his Differential Equations course (Michaelmas 2009) as the basis for these notes.

Contents

Prerequisites

A grasp of standard calculus (integration and differentiation).

1 Non-Rigorous Background

This course is about the study of differential equations, in which variables are investigated in terms of rates of change (not just with respect to time). It is, obviously, an area of mathematics with many direct applications to physics, including mechanics and so on. As such, it is important to have a grasp of how we codify a physical problem; we introduce this with an example:

Proposition 1.1 (Newton's Law of Cooling)**.** *If a body of temperature $T(t)$ is placed in an environment of temperature T_0 then it will cool at a rate proportional to the difference in temperature.*

> **Definition 1.2.** A *dependent variable* is a variables considered as changing as a consequence of changes in other variables, which are called *independent variables.*

In the example of Newton's Law of Cooling, the dependent variable is the temperature T which *depends* upon the independent variable time, t. The standard (Leibniz) notation for differentiation then gives us these equivalent forms for Newton's Law:

$$\begin{aligned} \frac{\mathrm{d}T}{\mathrm{d}t} &\propto T - T_0 \\ \frac{\mathrm{d}T}{\mathrm{d}t} &= -k\left(T - T_0\right) \end{aligned}$$

where we take k to be a constant; in fact, we require the constant of proportionality $k > 0$ for actual physical temperature exchanges.

Having established this basic approach, we shall begin with a fairly informal overview of differentiation and integration, to help us understand the techniques we will develop later. For a fully rigorous (axiomatic) approach to calculus, see the Analysis courses.

1.1 Differentiation using Big O and Little-o Notation

We define the rate of change of a function $f(x)$ as being

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

which is pictorially equivalent to the gradient of f at x.

Note that the limit can be taken from above or below, written $\lim_{h \to 0^{\pm}} \frac{f(x+h)-f(x)}{h}$, with both side limits being equal for differentiable functions. (Hence $f(x) = |x|$ is not differentiable at $x = 0$.)

We use various notations, given $f = f(x)$:

$$\frac{\mathrm{d}f}{\mathrm{d}x} \equiv f'(x) \equiv \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)[f(x)] \equiv \frac{\mathrm{d}}{\mathrm{d}x} f$$

where $\frac{\mathrm{d}}{\mathrm{d}x}$ is a *differential operator*. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right) \equiv \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \equiv f''(x)$$

To try and come up with a concise and useful way of writing f in terms of $\frac{\mathrm{d}f}{\mathrm{d}x}$, we introduce another notation (or two).

Definition 1.3. We write

$$f(x) = o(g(x))$$

as $x \to c$ if

$$\lim_{x \to c} \frac{f(x)}{g(x)} = 0$$

and we say f is *little-o* of g (as x tends to c).

This definition allows us to make explicit what we mean by f 'grows more slowly' than g.

Example 1.4.

(i) $x = o(\sqrt{x})$ as $x \to 0^+$.

(ii) $\ln x = o(x)$ as $x \to +\infty$.

Definition 1.5. We say that

$$f(x) = O(g(x))$$

as $x \to c$ if

$$\frac{f(x)}{g(x)}$$

is bounded as $x \to c$, and we say f is *big-O* of g (as x tends to c).

This similarly gives a rigorous definition of what it means to say that g 'grows at least as quickly' as f. Indeed, if $f = o(g)$ then it follows that $f = O(g)$.

Example 1.6. $\frac{2x^2-x}{x^2+1} = O(1)$ as $x \to \infty$. Similarly, $2x^2 - x = O\left(x^2+1\right) = O\left(x^2\right)$.

It follows that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{f(x+h) - f(x)}{h} + \frac{o(h)}{h}$$

the term on the right being referred to as the *error term*. (If we wrote it as $\epsilon(h) = \frac{o(h)}{h}$ we would have a function ϵ such that $\frac{\epsilon}{h} \to 0$ as $h \to 0$.)

Thus

$$h\frac{\mathrm{d}f}{\mathrm{d}x} = f(x+h) - f(x) + o(h)$$

and hence

$$f(x+h) = \underbrace{f(x) + h\frac{\mathrm{d}f}{\mathrm{d}x}}_{\text{linear approximation}} + o(h)$$